BRL CR 106

BRL

CONTRACT REPORT NO. 106

WSU SDL 72-01

THEORY OF EQUATIONS OF STATE:
ELASTIC-PLASTIC EFFECTS II

Prepared by

Washington State University
Pullman, Washington

May 1973

USA BALLISTIC RESEARCH LABORATORIES
ABERDEEN PROVING GROUND, MARYLAND

BRL CONTRACT REPORT NO. 106

FINAL REPORT

THEORY OF EQUATIONS OF STATE:

ELASTIC-PLASTIC EFFECTS II

by

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ABSTRACT

After a brief review of the history of material failure studies and of measurements of stored energy in plastically deformed bodies, energy and stress and strain relations are developed for both elastic and plastic deformation in uniaxial strain. The physical effects of plastic deformation are discussed and foundations are laid for thermodynamic calculations of plastically deformed materials. Calculations of both thermodynamic and mechanical parameters are described for uniaxial strain with and without energy storage in internal strains. Conditions of uniaxial strain are found to yield new thermodynamic relations which are useful for synthesizing constitutive relations.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

W	= work of uniaxial compression
E	= internal energy per unit mass
S_j	= j^{th} principal component of stress deviator
e	= stress deviator
S	= entropy per unit mass
τ	= resolved shear stress
p_j	= j^{th} principal component of compressive stress
dQ	= heat transferred in a thermodynamic process
V	= volume per unit mass
θ	= volume dilatation
ϵ	= strain
x,y,z	= Cartesian coordinates
R_D	= ratio of work of deformation to compressive work
K	= bulk modulus
λ	= Lamé constant
μ	= Lamé constant
σ	= stress
ν	= Poisson's ratio
T	= temperature, °K
A	= Helmholtz free energy
C_V	= specific heat at constant volume
γ	= resolved shear strain
r	= energy stored in dislocation
N	= dislocation length per unit volume
a	= dislocation multiplication parameter
b	= dislocation multiplication parameter
m	= dislocation multiplication parameter

t   = time

δ   = distance between imperfections which cause multiplication

n   = a characteristic frequency associated with pinned dislocations

U   = activation energy

V_δ = shear wave velocity

k   = Boltzmann's constant

I. INTRODUCTION

From the earliest times men have been concerned with the conditions of failure of structural materials. Little is known of early Egyptian, Greek and Roman knowledge of the subject, though it is evident from their extensive constructions that they had some effective working rules. During the Renaissance there began to be recorded systematic attempts to study and develop the concepts of material failure. Leonardo da Vinci describes an apparatus for testing the strength of iron wire. In a note on "Testing the Strength of Iron Wires of Various Lengths" he says, "...The object of this test is to find the load an iron wire can carry. Attach an iron wire two braccia long to something that will firmly support it, then attach a basket or any similar container to the wire and feed into the basket some fine sand through a small hole placed at the end of a hopper. A spring is fixed so it will close the hole as soon as the wire breaks. The basket is not upset while falling, since it falls through a very short distance. The weight of sand and the location of the fracture of the wire are to be recorded. The test is repeated several times to check the results. Then a wire of one-half the previous length is tested and the additional weight it carries is recorded; then a wire of one-fourth length is tested and so forth, noting each time the ultimate strength and the location of the fracture." (Timoshenko, 1953) It is particularly interesting to note that even here, in the fifteenth century, the concepts of graduated tests, systematic recording of observations, reproducibility, and variation of a single parameter while holding others constant were fully developed.

Many of the most famous names in science were involved with problems of material failure: Galileo, Euler, Lagrange, Coulomb, Poisson and Kelvin, to name but a few. The basic concepts of plasticity were developed by Barré Saint-Venant and described in several papers during the latter half of the nineteenth century.

1

This was a period of intensive activity in mechanics and one subject of interest was the search for a suitable criterion for failure of ductile materials. James Clerk Maxwell described the distortion energy concept of failure in a letter to Lord Kelvin in 1856. Although criteria based on this concept are widely accepted today, it did not appear in print until the early part of this century, when it was developed independently by M. T. Huber and R. von Mises.

Since 1934 when E. Orowan, M. Polanyi and G. I. Taylor independently developed the concept, and especially since 1950, failure theories of ductile solids have been based primarily on dislocation behavior. A great body of information on dislocations and their properties and behavior has grown up during the last twenty years, and it is realized that the continuum concepts of failure are, in a microscopic sense, ill founded. It is not possible to displace them, however, because the theory of dislocations is incomplete, because it is much more complicated than continuum theories, because continuum theories work quite well, and because there have been inadequate and only partially effective efforts to bridge the chasm which separates the microscopic behavior of dislocations and the macroscopic behavior of real materials.

Because of this gap between microscopic understanding and macroscopic behavior, there has been an intensive parallel effort to develop and refine continuum concepts of plasticity and yield during recent years.

Until the 1940's there was little concern about the thermodynamics of plastic strain. In 1948 Carl Eckart published an important paper on the thermodynamics of anelastic strain, and Tolman and Fine discussed applications of the Onsager theory of irreversible thermodynamics to problems of plasticity. These were followed by Bridgman's paper of 1950, an article by Reiner in 1958, and then by a spate of articles in the 1960's, of which perhaps the best known are those by B. Coleman and his coworkers, Meixner, and by G. A. Kluitenberg.

Except for Bridgman's paper, most of these have been concerned with general and quite abstract formulations. Indeed it is quite often very difficult to ferret out the underlying physical assumptions on which the work is based.

There appear, then, to be two needs for current work. One is to formulate explicitly and sharply physical foundations for a thermodynamic theory of plasticity which can be grasped by other than trained mathematicians and then to apply these to a relatively simple yet experimentally realistic physical situation. The other is to join the continuum thermodynamic-mechanics developments to dislocation theory in a manner which may be useful to both subjects.

In the present report some progress is made toward each of these goals. Further, since the question of stored energy in plastic work is everpresent and basic to the subject, a brief review of the state of stored energy measurements is given.

The principal results of the theoretical development are that (1) a point of view is developed that assumes the thermodynamics of plastic deformation to be essentially reversible if the correct variables are chosen; (2) theoretical calculations and experiments in one-dimensional strain introduce density as a variable, and this leads to phenomena and thermodynamic relations whose utility goes beyond that found for conditions of uniaxial stress; (3) a combination of thermodynamics and dislocation theory is fruitful for both fields, but the required relations are not completely developed here.

II. EXPERIMENTAL MEASUREMENTS OF STORED ENERGY OF DEFORMATION

Methods for measuring the stored energy in a plastically deformed solid may be put into two categories:

(1) single-step methods in which all the necessary measurements are made during the deformation;

(2) two-step methods in which the stored energy is measured following deformation.

2.1 Single-Step Methods

These methods are based on a direct application of the first law of thermodynamics. The stored energy E_S is the algebraic sum of the work, W, and the heat, Q. Work may be evaluated from stress and strain measurements. The heat effect is determined either from temperature rise of the specimen deformed under adiabatic conditions or by calorimetry.

The first successful attempt to measure the stored energy in a material undergoing a deformation was made by Hort (1906, 1907). In his experiments, Hort conducted tensile tests in iron in a water calorimeter. The calorimeter was calibrated by electric heating and thermoelastic cooling. For true strains ranging from 0.034 - 0.109, the measurement of stored energy in iron gave values of 2.2 to 5.1 cal/g atom.

Farren and Taylor (1925) determined the heat evolution associated with nearly adiabatic extension of polycrystalline specimens of steel, copper, and aluminum, and of a single crystal of aluminum. The samples were pulled for a short time and work was obtained from a recorded stress-strain curve. The heat evolved was computed from temperature rise in the specimen measured with a thermocouple placed inside an axial hole in the specimen. In 1934, Taylor and Quinney repeated these measurements by deforming steel and copper under torsion to obtain higher strains. Mechanical work was evaluated as in their previous work while

heat was measured both by a thermocouple and by a water calorimeter. The two methods gave similar results.

Maier and Anderson (1934) were the first to measure energy stored during wire drawing by the single step method. Their working metals were aluminum and copper.

The measurement of stored energy under compression was initiated by Kunin and Senilov (1936). They compressed copper by stages in a testing machine; work was measured by special "crusher" gauges. Epifanov and Rebinder (1949) determined the energy stored in chips of aluminum formed by drilling specimens inside a calorimeter. The difference between the energy input to the drill and heat evolved was considered to be the energy stored in the chips.

Eugene (1953, 1954) determined stored energies in lead, copper, and steel by using a freely falling weight to deform the specimens. He obtained stored energy from the difference between the known energy of the weight striking the specimen and the heat effects as measured by its temperature rise. He was able to find very small amounts of stored energy i.e. less than 1% compared to the average of around 10% reported by earlier workers.

Williams (1960) developed a device in which specimens were deformed by freely falling hammers of tungsten carbide. Since the period of contact is less than 2×10^{-3} sec, the deformation induced in the sample is assumed to be adiabatic. The sample is supported by a single thermocouple which measures the temperature rise. The work term is difficult to evaluate because of the mechanical losses due to suspension of the hammers. Heat transfer between the hammers and the sample is small because of the short interval of contact between them. Heat loss due to interaction with the surroundings is minimized by performing the experiments in vacuum. Williams (1961, 1962, 1963) has determined the stored energy in copper and copper-silver alloy by this method.

All of the measurements described above were carried out at room temperature.

Erdmann and Jahoda (1964) designed a device for deforming metal wires at low temperatures and for simultaneously measuring thermal conductivity, electrical conductivity, and heat liberated by the deformation. The experiment is beset with the usual difficulties associated with low temperature experiments which leads to some uncertainty in data reduction.

Williams (1963) also developed yet another apparatus in which the heat developed during the deformation of a sample under tension is measured by the amount of vaporization of a working fluid. The method is generally applicable from room temperature to 77°K. Wolfenden and Appleton (1963) have developed a method similar to the calorimetry of Williams. The fluid used is liquid nitrogen. These authors determined the stored energy in both single and polycrystals of copper and aluminum.

O. W. Dillon (1962, 1966, 1968) in a series of papers determined the heat generated during torsional oscillations of aluminum and copper tubes and the heat generated during the propagation of plastic waves in aluminum. This apparatus is based on the design of Farren and Taylor (1925).

2.2 Two-Step Methods

In these methods a material is deformed plastically by any suitable process. The energy spent in deformation may or may not be measured. The stored energy is determined in a subsequent step by measuring the difference in the thermal behavior of the deformed and a reference specimen in attaining identical final states. The difference in these heat effects is the stored energy of cold work.

Two step methods may be divided into two general classes.

(1) Annealing methods.

(2) Reaction methods.

(1) Annealing methods may be further subdivided into:

(a) Isothermal annealing, where the stored energy is released at a constant temperature. The energy release may be determined either as a function of time or as a total quantity after a chosen interval of annealing time.

(b) Anisothermal annealing, where the temperature is increased at a set rate, and the released energy is measured as a function of temperature.

Annealing methods have the added advantage of providing information on the kinetics of the release of stored energy.

(2) Reaction method. In this method the plastically deformed and standard specimens of a material are allowed to react with a molten substance in a calorimeter. The difference between the heats of reaction of standard and deformed specimens is considered to be the stored energy of plastic deformation.

Isothermal Annealing Method

Borelius and his coworkers (1953) seem to have been the first to measure stored energy by this method. They used an isothermal jacket microcalorimeter to investigate the energy evolved at 100°C by aluminum, copper, and zinc immediately after rolling and the energy released at 60°C by aluminum and copper after a lapse of 41 hours and 45 hours respectively. The calorimeter consisted of a vapor thermostat, which provided an isothermal environment for a chamber containing the specimen. The temperature was measured by 30 junctions of sensitive differential thermopile, one set of which was in contact with the chamber. The other set was in contact with the specimen. In operation, the sample was preheated and was rapidly placed in position. The release of heat from the specimen was indicated by the temperature difference across the pile.

Åström (1956, 1964) used the same equipment to study the release of stored energy from aluminum and molybdenum plastically deformed under compression. Gordon (1954, 1955) also used the Borelius et al. type of calorimeter and method to determine energy stored and the kinetics of isothermal release in the range 150°C - 220°C in copper specimens deformed under tension. Bailey and Hirsch (1960) have measured stored energy in silver by this means.

Anisothermal Annealing Methods

There are two basic methods which may be used to obtain results under anisothermal annealing. Both methods are normally used in a differential arrangement where the cold worked sample is compared to an annealed sample. In one case, the "adiabatic method," both samples are heated internally so as to minimize the transfer of heat between samples and the evacuated furnace, and the stored energy is given by the difference between powers required to heat the two samples. In the second method the heat required to raise the temperature of the samples is determined from the transfer of heat from the furnace. The two specimens are placed so as to minimize the thermal interaction between them. This method is called "differential thermal analysis."

The adiabatic method was first used by Quinney and Taylor (1937) to determine stored energies in copper, 70 copper - 30 zinc alloy, aluminum, iron and nickel. Welber (1952) and Clarebrough et al. (1952) built equipment based on the design of Quinney and Taylor. Clarebrough and his coworkers have measured the release of energy stored in nickel and copper deformed by various processes. A summary of these investigations has been given by Boas (1957).

The differential thermal analysis annealing method was first used by Satô (1931). He obtained differential heating curves for cold worked specimens of 70 Cu - 30 Zn, silver, copper, aluminum, iron, nickel and brasses. He annealed standard specimens of identical materials placed side by side, but not

8

in contact with, a block of silver. The block was heated at the rate of 5°C/min to the temperature at which the standard had been annealed. The heating was repeated without moving the specimens to evaluate the corrections necessary to account for differences between the two specimens not due to cold work.

Henderson and Koehler (1956) have used the technique to study energy release in deformed copper and brass at 77°k. These materials were deformed under compression and the energy release was measured from 77°k to 300°k. Recently Van den Beukel (1963) has repeated Henderson and Koehler's work and has also studied the energy release in deformed nickel, silver, and gold.

Reaction Methods

Reactions which have been used for measuring stored energy of plastic work are dissolution in aqueous solvents and oxidation, and dissolution in liquid tin. In order to measure the energy difference between a deformed and an undeformed sample by measuring the difference between the heats of solution in any substance, it is necessary that the heat of solution be small, the solubility large and the rate of solution at a moderate temperature very rapid. The only solvents which can thus qualify are liquid metals. The application of this method, though started in 1929 by Smith (1929), has been applied with success starting with the works of Bever (1951) and his associates. The method they currently use is given by Howlett et al. (1962). To date, measurements by reaction methods have been limited to copper-gold and silver-gold alloys.

Some other calorimeters which may be used to study release of deformation energy are those constructed by Stansbury et al. (1959), Sherwin (1964), Arndt and Fujita (1963), and Calvet and Prat (1963).

9

2.3 Status of Stored Energy Measurements and Current Problems

The literature now contains a considerable number of investigations in which the stored energy of plastic deformation has been measured. The distribution of published papers in time between 1900 and 1965 is shown in Figure 1. However, it is still difficult to evaluate critically the relative accuracy of the various methods described above. There are at least three reasons for this situation.

(1) Scarcity of data for any metal which can be used as a standard. Polycrystalline copper comes closest.

(2) Reporting only of reproducibility and not accuracy of results by investigators. This is not adequate since accuracy is indicated by reproducibility only in the absence of systematic errors.

(3) The exact agreement between different methods should be possible only under certain conditions. These conditions concern the characteristics of calorimeters, thermocouple behavior, thermoelastic effects, the process by which deformation is induced in a specimen, etc. However, within limitations imposed by the above three factors, some comparisons are possible; these have been made by Williams (1967) in a recent review article.

An appreciation of the basis for limited comparisons among experimental results may be obtained by simply listing variables which affect the quantity of stored energy in a cold worked material.

A. Material as a variable
   1. Purity
   2. Composition of solid solution alloys
   3. Crystallinity, i.e. single crystal or polycrystal
   4. If polycrystalline, the effect of grain size and grain structure
   5. Dislocation characteristics, vacancies, etc.

B. Process of deformation as a variables
   1. Extension
   2. Compression
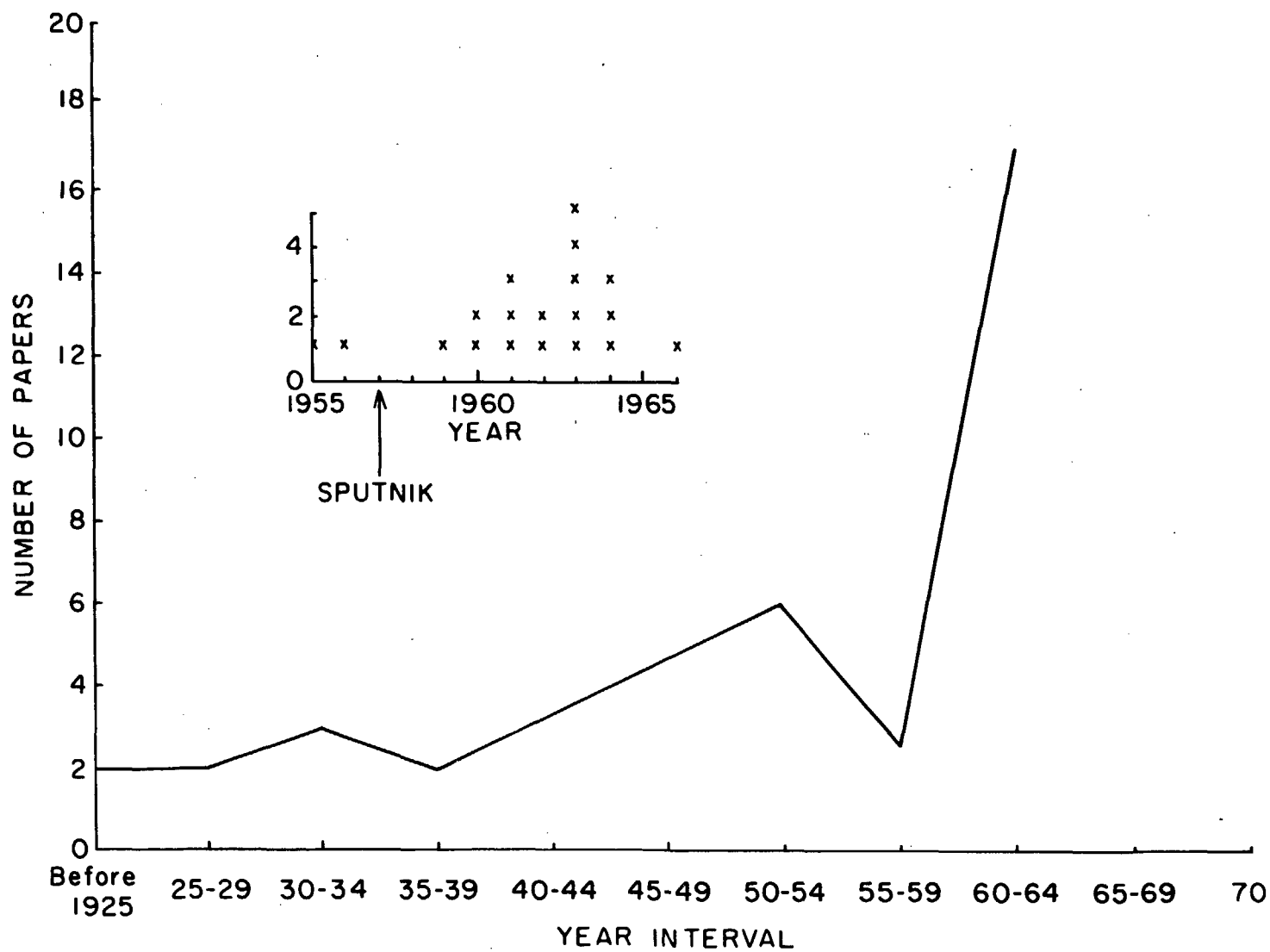
Fig. 1. Time Distribution of Papers on Stored Energy Measurement in Deformed Materials

3. Torsion
4. Rolling
5. Wire drawing
6. Cutting processes such as filing, drilling, etc.
7. Deformation in fatigue, i.e. cyclic loading
8. Degree and rate of deformation

C. Effects of temperature on cold work.

Present activity in the field is directed toward evolving a model or models which will explain the mechanisms of energy storage and release. The present interpretation of energy storage is based on ideas of the production of inhomogeneous lattice strains and the generation of imperfections like dislocations, point defects, stacking faults in hexagonal materials, twins and reduction of order in alloys. However, a quantitative interpretation of stored energy processes remains uncertain primarily due to lack of a precise accepted definition of cold work and to the degree of inaccuracy in measurement of stored energy.

It will be seen from the above description that the stored energy of cold work has been measured exclusively in metals. Even among metals, copper, nickel, aluminum, silver and iron, have been most commonly investigated; infrequent or solitary measurements of stored energy of cold work in zinc, lead tin do exist. The stored energy of cold work has also been investigated in a few alloys, notably in Cu-Ag, Cu-Au, and Cu-Ni. The absence of any such investigation in ionic compounds is striking considering the large amount of work done in LiF, MgO and other ionic solids (see for example, (Gilman, 1969)).

Laboratories where investigations on the stored energy of cold work and its release mechanism are being carried out at present are few. The prominent laboratories and persons involved in these measurements are:

(1)   Oak Ridge National Laboratory, Oak Ridge, Tenn. R. O. Williams, A. Wolfenden, and R. A. Vandermeer.

12

(2)   Dept. of Metallurgy, MIT, Cambridge, Mass. M. B. Bever and A. L. Titchener

(3)   Dept. of Metallurgical Engineering, IIT, Illinois, P. Gordon

(4)   Cavendish Laboratory, Cambridge, U.K., J. E. Bailey and P. B. Hirsch

(5)   Dept. of Metallurgy, Univ. of Liverpool, U.K., A. S. Appleton

(6)   Div. of Tribophysics, C.S.I.R.O., Univ. of Melbourne, Australia,

        L. M. Clarebrough, M. E. Hargreaves, M. H. Loretto and A. K. Head

Present ideas about the mechanisms of storage of cold work energy and its release are given in what follows.

The small percentage (5 to 10%) of the energy expended in plastically deforming a metal, which is stored in the deformed metal, is associated with defects generated during the deformation. A significant part of the stored energy is associated with the distribution and density of dislocations in the bulk of the deformed metal. The remaining part of the stored energy is associated with imperfections like point defects, vacancies, twins, stacking faults, etc. However, this partition of stored energy among the various imperfections as well as the absolute value of the stored energy is dependent on purity, degree of deformation, temperature of deformation, and grain size. In general, the energy stored during plastic deformation of a metal increases with the degree of deformation, and may or may not reach a saturation before fracturing. Also for a given extent of deformation, the value of stored energy increases with decreasing temperature of deformation of a metal. In general, the higher the purity of a metal, the lower the stored energy of cold work. And lastly, for metals, stored energy for a given deformation increases with decrease in grain size for small strains (≤ 0.20) and then becomes relatively independent of grain size for large strains.

The inference about apportionment of stored energy among the various imperfections in a metal is made from the correlation of release of stored

energy with changes in other physical properties, e.g., changes in electrical resistivity, density and with structural studies using conventional metallographic techniques and electron transmission microscopy.

The release of stored energy proceeds in two or more stages. However the various processes by which the annealed state is reached are classified into two categories -- recovery and recrystallization. The apportionment of the released energy into these two classes is based on the definition that recovery encompasses all those annealing phenomena which precede recrystallization in any local region of the material. On the other hand, recrystallization is defined as the appearance of comparatively strain-free grains which are separated from the deformed matrix or from other recrystallized grains by high-angle grain boundaries and alternatively, recrystallization may be defined as the appearance of virtually strain-free grains which have grown in the deformed matrix by the movement of large angle grain boundaries.

A common feature in all the investigations on metals and alloys is that recrystallization is accompanied by a marked release of stored energy as well as a decrease in electrical resistivity and an increase in density. Electron microscope observations indicate that this release of energy and concomittant changes in the electrical resistivity and density are associated with a change in the distribution and density of dislocations. Hence the magnitude of energy released during recrystallization can be associated with the energy of the dislocations. Bailey and Hirsch (1960) seem to have been the first to establish such an association, which they did for polycrystalline silver. However, Clarebrough et al. (1961) disputed the conclusion of Bailey and Hirsch that theory of flow stress involving piled up groups of dislocations is not applicable to silver.

For recovery the experimental situation is not so well defined. The ratio of stored energy released during recovery to total stored energy varies between 0.03 to 0.7 depending on the purity of material, strain, and temperature

of deformation. Evaluations of changes in other properties accompanying recovery in metals deformed at room temperature have been aided insignificantly by structural observations. For metals deformed at the temperature of liquid nitrogen the energy stored is four times that for deformation at room temperature. The annealing of point defects is assumed to contribute to the large releases of energy associated with recovery below room temperature, in addition to the contribution of other imperfections. Bailey and Hirsch (1960) could not detect any change in the dislocation density and distribution before and after recovery in silver, although during the recovery half the total stored energy was liberated.

In alloys a major source of energy release during recovery may be associated with the return of short range order destroyed by deformation.

To sum up, existing experimental measurements on the stored energy of cold work and the release of the energy with concomittant variation in the physical structural properties in the most extensively studied metal, i.e. copper, is not sufficient to test validity of the two existing theories of work hardening of pure metals. In one theory, due to Friedel (1955) and Seeger et al. (1957), the flow stress is determined by long-range stresses due to groups of piled up dislocations. In the second theory, due to Cottrell (1953), Basinski (1959), and Hirsch (1958), the flow stress is determined by short-range interactions between crossing dislocations.

III. PLASTIC DEFORMATION IN UNIAXIAL STRAIN

3.1 Definitions and Basic Relations

The uniaxial strain condition applies to plane wave compression and rarefaction. If wave propagation is in the x-direction, the work of uniaxial compression is

$$dW = -Vp_x \, d\varepsilon_x = -p_x \, dV \tag{1}$$

where $p_x = -\sigma_x$ = compressive stress in the x-direction and $d\varepsilon_x \equiv dV/V$. Principal axes of stress and strain lie in and normal to the direction of propagation. If E is internal energy of unit mass,

$$dE = dQ - Vp_x \, d\varepsilon_x \tag{2}$$

where dQ is heat communicated to the mass element by conduction or other means. The work term of Eq. (1) can be expressed as work of compression plus work of deformation by introducing stress and strain deviators, S_j and e_j:

$$dW = -Vp \, d\theta + V \sum_j S_j \, de_j \tag{3}$$

where

$$S_j = -p_j + p \qquad \qquad \left. \begin{array}{l} \\ \end{array} \right) \tag{4a}$$
$$\left. \begin{array}{l} \end{array} \right) \quad j = x, y, z.$$
$$e_j = \varepsilon_j - \theta/3 \qquad \qquad) \tag{4b}$$

$$S_x = 4\tau/3, \quad S_y = S_z = -2\tau/3 \tag{5a}$$

$$\tau = -(p_x - p_y)/2 \tag{5b}$$

$$e_x = 2\varepsilon_x/3, \quad e_y = e_z = -\varepsilon_x/3, \quad d\theta = d\varepsilon_x. \tag{5c}$$

Using these relations, Eq. (3) becomes

$$dW = -Vpd\epsilon_x + V[(4\tau/3) \cdot (2/3)d\epsilon_x - 2(2\tau/3)(-d\epsilon_x/3)]$$

$$= -Vpd\epsilon_x + 4V\tau d\epsilon_x/3 \tag{6}$$

$$= -Vpd\epsilon_x (1 - 4\tau/3p) \tag{7}$$

$$= -Vpd\epsilon_x (1 - R_D) \tag{8}$$

where R_D is the ratio of work of deformation to compressive work. For elastic changes the constitutive relations can be written in incremental form to account for changes in the elastic moduli with strain:

$$dp = -Kd\theta = -Kd\epsilon_x \tag{9}$$

$$dS_j = 2\mu de_j \tag{10}$$

or

$$d\sigma_x = (\lambda + 2\mu)d\epsilon_x \tag{11}$$

$$d\sigma_y = \lambda d\epsilon_x = d\sigma_z \tag{12}$$

$$d\tau = \mu d\epsilon_x \tag{13}$$

where K is bulk modulus and λ, μ are the Lamé constants. Values of K, λ, μ depend on whether strain occurs at constant temperature or constant entropy. Isothermal moduli will be denoted K^T, λ^T, μ^T; isentropic moduli K^S, λ^S, μ^S. When the superscript is omitted the meaning is obvious or the question is indifferent.

We can invoke the condition of uniaxial strain to obtain:

$$dp_y = -d\sigma_y = \nu dp_x/(1-\nu) = dp_z \tag{14}$$

$$dp = (dp_x/3)(1+\nu)/(1-\nu) \tag{15}$$

$$d\tau = -(dp_x/2)(1-2\nu)/(1-\nu) \tag{16}$$

where $\nu = (\lambda/2)/(\lambda+\mu) \equiv$ Poisson's ratio, which may depend on strain and temperature or entropy.

For small strains, Eqs. (11) - (13) can be integrated assuming moduli to be constant:

$$\sigma_x = (\lambda + 2\mu) \, \varepsilon_x \tag{17}$$

$$\sigma_y = \lambda\varepsilon_x = \sigma_z \tag{18}$$

$$\tau = \mu\varepsilon_x \tag{19}$$

$$\varepsilon_x = (V - V_0)/V_0 \tag{20}$$

where V is specific volume and V_0 denotes a reference, stress and strain-free state. For this case

$$R_D = 2(1 - 2\nu)/(1 + \nu) \tag{21}$$

(see Eq. (8)). The dependence of R_D on ν is shown in Fig. 2.

This discussion of elastic strain provides a framework within which questions of plasticity may be discussed and it is important to note that the work of plastic deformation is bounded by the work of volume compression and that of elastic deformation. This is illustrated in Fig. 3 where curves of hydrostatic and uniaxial compression under adiabatic conditions are compared. For elastic strain, adiabatic conditions are also isentropic, so the slopes of OM and ON are $-(V_0/V)K^S$ and $(-V_0/V) (\lambda^S + 2\mu^S)$, respectively. Since $p_x = p - 4\tau/3$, the work of deformation is

$$W_d = -\int p_x dV + \int p dV = (4/3) \int \tau \, dV \tag{22}$$

which is just V_0 times the area of the sliver between ON and OM of Fig. 3. As noted in Fig. 2, this may be small or large depending on the value of ν.

When the strain cycle is completed by allowing the material to expand back to p or $p_x = 0$, the elastically strained sample under adiabatic conditions simply returns to the point 0 along the curve of compression, OA or OB. The

Figure 2

R_D = Work of Deformation/Work of Volume Compression
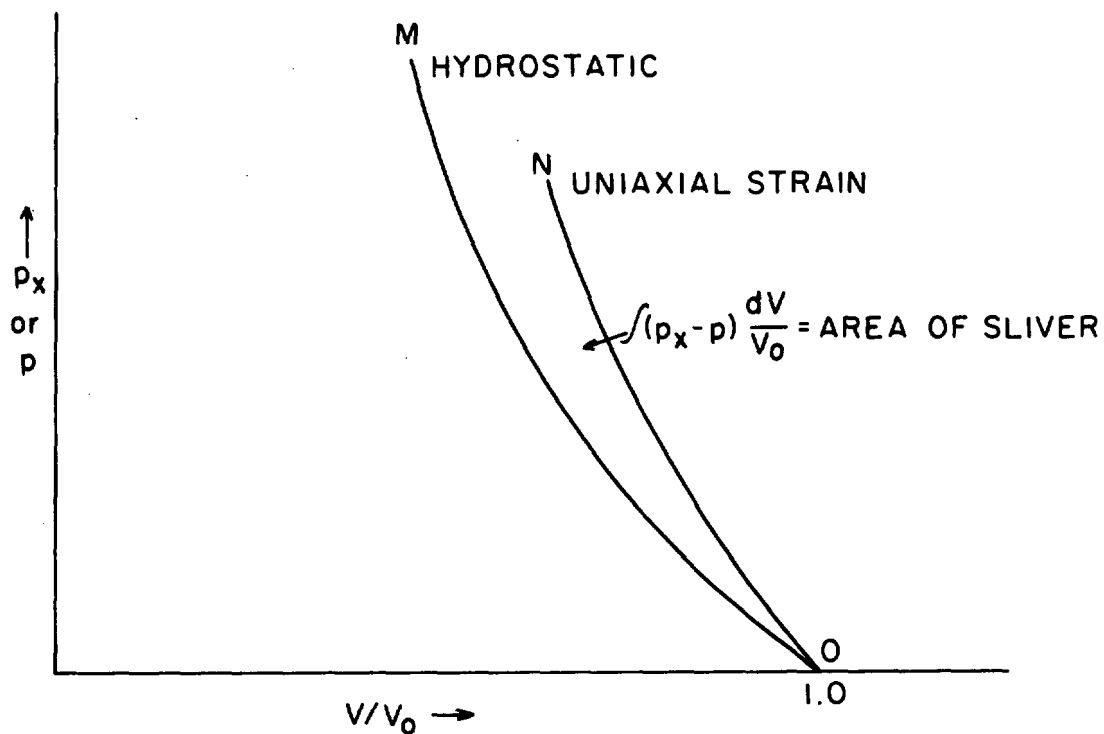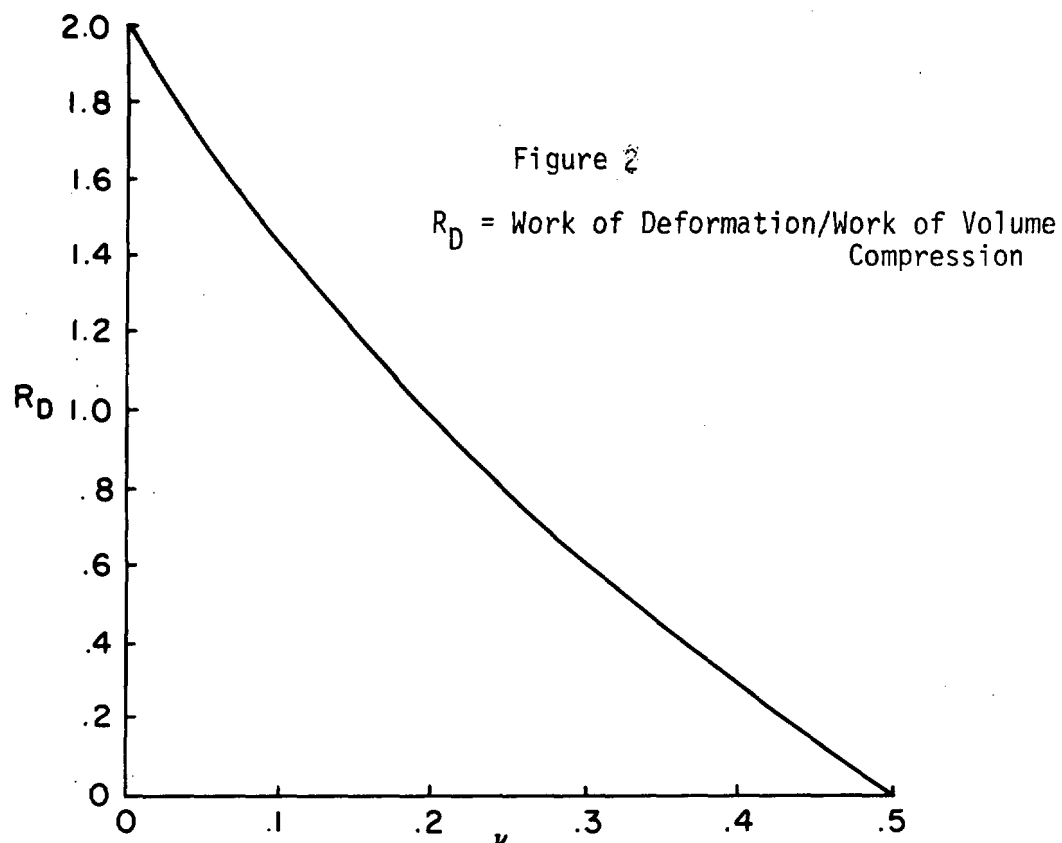
Figure 3   Adiabatic elastic compression of an isotropic solid in uniaxial strain and hydrostatically.

19

plastically deforming material, on the other hand, completes part of a hysteresis loop as shown in Fig. 4. The sections of this loop are formed according to the following process. Starting at point O the sample is compressed uniaxially until it starts to deform plastically at A. Along AB the plastic deformation is continuous until the maximum value of p_x is reached. At B the applied stress is allowed to start decreasing, $\sigma_x - \sigma_y$ is reduced and expansion occurs elastically. At C, $\sigma_y - \sigma_x$ has once more reached the yield value and plastic deformation continues from C to D. There are now two questions to be considered:

(i) What is the amount of plastic work done along the sections AB and CD? (Q1)

(ii) Can a thermodynamic state be defined at each point? (Q2)

We consider these questions in the order stated.
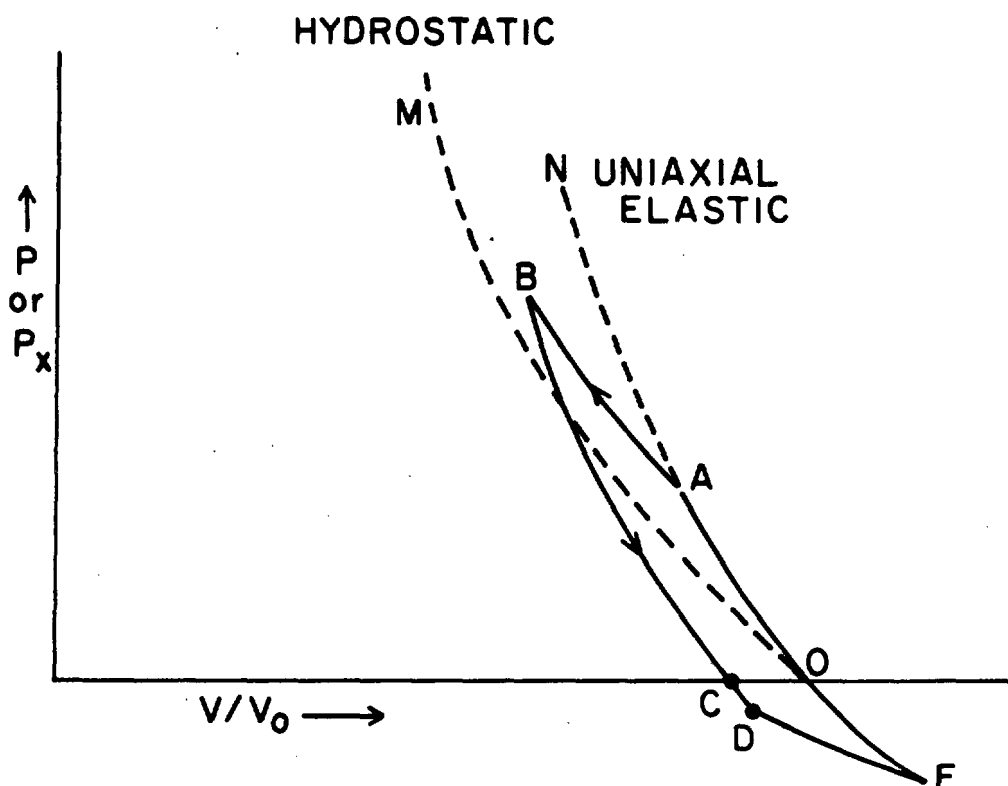
Figure 4 Adiabatic compression and expansion of an elastic-
plastic solid in uniaxial strain, OABCD.

3.2 Computation of the Plastic Work

An important convention in the theory of plasticity is the assumption that when plastic and elastic strain are occurring simultaneously, each increment in total strain can be decomposed into a sum of plastic and elastic increments, i.e.

$$d\epsilon = d\epsilon^e + d\epsilon^p \tag{23}$$

where superscripts "e" and "p" stand for "elastic" and "plastic."

This convention is discussed in Appendix A. It is also common to assume that there is no plastic contribution to the density change, so that

$$d\epsilon_j = d\epsilon_j^e + d\epsilon_j^p = de_j + d\theta/3$$

$$d\theta \equiv d\theta^e; \qquad d\theta^p = 0 \tag{24a}$$

$$de_j = de_j^e + de_j^p \tag{24b}$$

Then increments in the work of plastic and elastic deformation are given, respectively, by

$$dW_{dp} = V \sum_j S_j \, de_j^p \qquad \text{plastic} \tag{25}$$

$$dW_{de} = V \sum_j S_j \, de_j^e \qquad \text{elastic} \tag{26}$$

Since dW_{dp} is the only plastic work in this process, i.e. volume compression is purely elastic, we write henceforth

$$dW_p \equiv dW_{dp}.$$

We also assume that stresses are supported only by the elastic strains,

$$dS_j = 2\mu \, de_j^e. \tag{27}$$

A physical basis for Eq. (27) is discussed in Appendix B.

The condition of uniaxial strain requires only that total strain ε_y and ε_z vanish, i.e.

$$d\varepsilon_y{}^p + d\varepsilon_y{}^e = 0 \tag{28a}$$

$$d\varepsilon_z{}^p + d\varepsilon_z{}^e = 0 . \tag{28b}$$

Because of symmetry about the x-axis

$$d\varepsilon_y{}^e = d\varepsilon_z{}^e . \tag{29}$$

From Eqs. (5a), (10) and (27)

$$dS_x = 4d\tau/3 = 2\mu\, de_x{}^e \tag{30}$$

$$dS_y = -2d\tau/3 = 2\mu\, de_y{}^e \tag{31}$$

$$dS_x - dS_y = 2d\tau = 2\mu\, d(e_x{}^e - e_y{}^e)$$

$$d\tau = 2\mu d\gamma^e \tag{32}$$

where $\qquad d\gamma^e \equiv d(e_x{}^e - e_y{}^e)/2 = (d\varepsilon_x{}^e - d\varepsilon_y{}^e)/2. \tag{33}$

From Eq. (6) the total incremental work of deformation is

$$dW_d = 4V\tau\, d\varepsilon_x/3. \tag{34}$$

From Eqs. (26) and (30) - (32) the incremental work of elastic deformation is

$$dW_{de} = 8V\tau\, d\gamma^e/3$$

$$= (4V\tau/3\mu)d\tau . \tag{35}$$

The incremental work of plastic deformation is obtained by subtracting Eq. (35) from (34):

$$dW_p = (4/3)\, V\tau\, (d\varepsilon_x - d\tau/\mu) \tag{36}$$

23

where $\tau = \pm Y/2$ depending on whether the process is compressive or expansive. $Y > 0$ is the yield stress as given by the von Mises condition.

Eq. (36) can be obtained directly from Eqs. (4b), (24a) and (25). It can also be written in the form of Eq. (C5) in Appendix C:

$$dW_p = 2V\tau \, d\epsilon_x^p \tag{37}$$

If $\gamma^p \equiv (e_x^p - e_y^p)/2$, Eq. (37) can also be written as

$$dW_p = (8/3)V\tau \, d\gamma^p \tag{38}$$

The increment in elastic work represented by Eq. (35) can easily be seen to be reversible:

$$dW_{de} = (2V/3\mu) \, d(\tau^2). \tag{39}$$

As $|\tau|$ goes up and down, so does W_{de}.

The plastic work is irreversible by definition,

$$dW_p \geq 0. \tag{40}$$

This is readily illustrated with reference to Fig. 4. During elastic compression along OA in Fig. 4, $d\epsilon_x^p = 0$, $\tau < 0$, $d\tau < 0$, $dW_p = 0$. Along AB, $\tau \leq 0$ and $d\epsilon_x < 0$. For a material which doesn't work-harden, $d\tau = 0$, so $dW_p > 0$. If work-hardening occurs, $d\tau < 0$, but $d\tau/\mu$ cannot exceed $d\epsilon_x$ in magnitude since $d\epsilon_x = d\tau/\mu$ when compression is elastic. So in that case, too, $dW_p > 0$. Along BC the material is expanding elastically, τ is changing from negative to positive sign, $d\epsilon_x^p = 0$ and $dW_p = 0$. Along CD, $\tau > 0$, $d\epsilon_x > 0$ and again $dW_p > 0$ from Eq. (36) by the same argument applied to the path AB. Also $d\epsilon_x^p > 0$ along CD.

Eq. (36) or (37) provides an answer to question 1, raised above. We now turn attention to Q2.

IV. THERMODYNAMIC STATES IN PLASTIC FLOW

The simplest observations of plastic yield tell us a great deal about the process. Bend a piece of iron wire back and forth between your fingers a few times. It gets hot and is still solid iron. The first result is obvious; the second is verified by chemical and metallurgical analysis. It is, in fact, very hard to detect any changes in the material, but we know that some have occurred since repeated bending breaks the wire, yet it does not break at the first bending. Property changes due to bending are more obvious in a piece of copper. Bending produces marked increases in bending resistance before it breaks.

In developing a thermodynamic theory to deal with such conditions we can assume either that plastic work changes the material or that it does not. It is clear from our simple experiment that the latter assumption is a fairly good one and we accept it for the moment and consider its consequences.

4.1 Plastic Work Converted Entirely to Heat

Even though this be true, there is a difficulty associated with plastic strain which is illustrated in Fig. 4. On reversal of stress at any point, say B, the stress-strain curve is not retraced, even though the process be isothermal. Yet if we persist in the reversal we can find a sequence of applied forces which will return the material to its initial state under this condition that plastic work produces no material changes. This is illustrated in Figs. 6 and 7, to be discussed farther on in this section. This is a kind of geometric irreversibility to be distinguished from irreversibility of material properties. It complicates description of thermodynamic processes but does not essentially change the postulate that plastic working has put the material in a thermodynamic state characterized by the added heat, ΔQ. The situation is somewhat like that of a fluid. If we deform a viscous fluid at constant density, we recognize that heating occurs, but we suppose the final state to be describable in terms of thermodynamic

parameters of the original fluid. The fact that particular molecules have different neighbors is considered inconsequential.

In the same way it is inconsequential that neighbors may have changed after plastic deformation; thermodynamic parameters are the same. It is, in a sense, unimportant that the macroscopic form of the material has changed and that this makes a difference in the external behavior. This difference is observable in the solid and not in the fluid because the solid atoms undergo elastic strain during deformation, whereas the fluid atoms do not.

With such considerations as foundation, we are led to a <u>Fundamental Assumption</u>:

> There is always an elastically strained state
> imbedded in every deformed state. This elastic
> state is inherently reversible and can be recovered
> by suitable processes.

This assumption implies that an internal energy, E, exists and that it is a function of the elastic strains, ε^e, entropy, S, and perhaps of internal or microscopic strain variables. A representation of the internal energy function can be obtained by noting that each increment of work on the system is the sum of an elastic and a plastic increment. By the Fundamental Assumption the elastic increment is recoverable, so increments in internal energy can be written as

$$dE = TdS + dW_e \qquad (41)$$

where dW_e depends on the elastic strain increments. According to the first law this increase in internal energy must equal the total work done on the system plus the heat absorbed from the surroundings:

$$TdS + dW_e = dW + dQ . \tag{42}$$

This leads directly to the <u>Gibbs relation</u> for the increase in entropy produced in the process:

$$dS = (dW_p + dQ) / T \tag{43}$$

where $dW_p = dW - dW_e$.

According to Eq. (41) and the foregoing assumptions, internal energy can be expressed in terms of entropy and the elastic strains. For example

$$E = E(S, \varepsilon_x^e, \varepsilon_y^e)$$

Changes in entropy result solely from the agitations and elastic displacement of atoms; there is no entropy directly associated with the plastic strain. Nonetheless an apparent dependence of state on plastic strains may exist because of the procedure required to separate dW into elastic and plastic parts. For example, from Eqs. (1) and (37) we find that

$$dW_e = dW - dW_p = V\sigma_x \, d\varepsilon_x - 2V\tau d\varepsilon_x^p$$

so that

$$E = E(S, \varepsilon_x, \varepsilon_x^p).$$

The appearance of ε_x^p in this potential is illusory. It does not imply a physical dependence of E on plastic strain. Failure to realize this fact has led to some confusion in other work.

It will be useful to tabulate various forms of the internal energy, but first we note that increments in the Helmholtz potential will be in the form

$$dA = -SdT + dW_e \tag{44}$$

and that the entropy and temperature are given by

$$S = -(\partial A/\partial T)_{\varepsilon^e} \tag{45}$$

$$T = (\partial E/\partial S)_{\varepsilon^e} \tag{46}$$

where subscript ε^e indicates that all elastic strains are to be held constant. Using the equations developed earlier, we have the following representations: From Eqs. (6) and (35),

$$dW_e = -Vpd\varepsilon_x + 8V\tau d\gamma^e/3 \tag{47}$$

$$E = E(S, \varepsilon_x, \gamma^e) \tag{48}$$

$$A = A(T, \varepsilon_x, \gamma^e). \tag{49}$$

Since $Vd\varepsilon_x = dV$, this is equivalent to a representation in which V and γ^e are the mechanical variables.

From Eqs. (1) and (37),

$$dW_e = dW - dW_p = V\sigma_x d\varepsilon_x - 2V\tau d\varepsilon_x^p \tag{50}$$

$$E = E(S, \varepsilon_x, \varepsilon_x^p) \tag{51}$$

$$A = A(T, \varepsilon_x, \varepsilon_x^p). \tag{52}$$

From Eqs. (1) and (38)

$$dW_e = dW - dW_p = V\sigma_x d\varepsilon_x - 8V\tau d\gamma^p/3 \tag{53}$$

$$E = E(S, \varepsilon_x, \gamma^p) \tag{54}$$

$$A = A(T, \varepsilon_x, \gamma^p). \tag{55}$$

A representation which shows the dependence of state functions on elastic strains and temperature exclusively is

$$dW_e = V(\sigma_x \, d\varepsilon_x^e + 2\sigma_y \, d\varepsilon_y^e) \tag{56}$$

so that

$$E = E(S, \varepsilon_x^e, \varepsilon_y^e) \tag{57}$$

$$A = A(T, \varepsilon_x^e, \varepsilon_y^e) \tag{58}$$

Since entropy is defined as

$$S = -\partial A/\partial T,$$

it depends on the same variables as A.

The functions E, A, S, etc. are state functions, so their differentials are exact and lead to various Maxwell-like relations and compatibility conditions which are later shown to be useful.

Some common thermodynamic observables take unusual forms when these representations are used. Specific heat is an example.

From Eq. (49),

$$C_V \equiv T(\partial S/\partial T)_{\varepsilon_x} = T(\partial S/\partial T)_{\varepsilon_x, \gamma^e} + T(\partial S/\partial \gamma^e)_{T, \varepsilon_x} (\partial \gamma^e/\partial T)_{\varepsilon_x} \tag{59a}$$

From Eq. (52),

$$C_V = T(\partial S/\partial T)_{\varepsilon_x} = T(\partial S/\partial T)_{\varepsilon_x, \varepsilon_x^p} + T(\partial S/\partial \varepsilon_x^p)_{\varepsilon_x, T} (\partial \varepsilon_x^p/\partial T)_x \tag{59b}$$

From Eq. (55),

$$C_V = T(\partial S/\partial T)_{\varepsilon_x} = T(\partial S/\partial T)_{\varepsilon_x, \gamma^p} + T(\partial S/\partial \gamma^p)_{\varepsilon_x, T} (\partial \gamma^p/\partial T)_{\varepsilon_x} \tag{59c}$$

For Eq. (56) C_V is not defined but can be obtained by invoking the condition

$$d\varepsilon = d\varepsilon_x^e + 2d\varepsilon_y^e, \tag{60}$$

which follows directly from Eq. (24a) and the symmetry of uniaxial strain. In this way one of the three forms of Eq. (59) will be obtained.

One may reasonably ask whether derivatives like $(\partial S/\partial \varepsilon_x^p)_{T,\varepsilon_x}$, which appears in Eq. (59b), are physically meaningful in view of the nature of plastic strain. Since plastic strain enters in a contrived way, as explained in the paragraph following Eq. (43), they can always be expressed in terms of more obviously thermodynamic derivatives. For example, calculate dS from Eqs. (52) and (58):

$$dS = (\partial S/\partial T)_{\varepsilon_x,\varepsilon_x^p} dT + (\partial S/\partial \varepsilon_x)_{T,\varepsilon_x^p} d\varepsilon_x$$

$$+ (\partial S/\partial \varepsilon_x^p)_{T,\varepsilon_x} d\varepsilon_x^p$$

$$dS = (\partial S/\partial T)_{\varepsilon_x^e,\varepsilon_y^e} dT + (\partial S/\partial \varepsilon_x^e)_{T,\varepsilon_y^e} d\varepsilon_x^e$$

$$+ (\partial S/\partial T)_{T,\varepsilon_x^e} d\varepsilon_y^e .$$

Since we wish to obtain $\partial S/\partial \varepsilon_x^p$ in terms of elastic strain derivatives, write $d\varepsilon_x^e$ and $d\varepsilon_y^e$ in terms of $d\varepsilon_x$ and $d\varepsilon_x^p$; i.e.

$$d\varepsilon_x^e = d\varepsilon_x - d\varepsilon_x^p$$

$$d\varepsilon_y^e = -d\varepsilon_y^p = d\varepsilon_x^p/2 .$$

· Then equating the two values of dS gives

$$0 = [(\partial S/\partial T)_{\varepsilon_x \varepsilon_x^p} - (\partial S/\partial T)_{\varepsilon_x^e \varepsilon_y^e}]dT + [(\partial S/\partial \varepsilon_x)_{T,\varepsilon_x^p}$$

$$-(\partial S/\partial \varepsilon_x^e)_{T,\varepsilon_y^e}]d\varepsilon_x + [(\partial S/\partial \varepsilon_x^p)_{T,\varepsilon_x}$$

$$-(1/2)(\partial S/\partial \varepsilon_y^e)_{T,\varepsilon_x^e} + (\partial S/\partial \varepsilon_x^e)_{T,\varepsilon_y^e}] \, d\varepsilon_x^p . \quad (61)$$

Specification of the deformation process requires specification of dT and two strain variables, so coefficients of dT, $d\varepsilon_x^e$, $d\varepsilon_x^p$ in Eq. (61) must vanish independently; therefore

$$(\partial S/\partial \varepsilon_x^p)_{T,\varepsilon_x} = (1/2) (\partial S/\varepsilon_y^e)_{T,\varepsilon_x^e} - (\partial S/\partial \varepsilon_x^e)_{T,\varepsilon_y^e} \quad (62)$$

From Eqs. (44) and (56)

$$dA = -SdT + V\sigma_x d\varepsilon_x^e + 2V\sigma_y d\varepsilon_y^e$$

$$S = -\partial A/\partial T; \quad V\sigma_x = \partial A/\partial \varepsilon_x^e$$

$$\partial S/\partial \varepsilon_x^e = -\partial^2 A/\partial T \partial \varepsilon_x^e = -(\partial V\sigma_x/\partial T)_{\varepsilon_x^e \varepsilon_y^e}$$

$$\partial S/\partial \varepsilon_y^e = -2(\partial V\sigma_y/\partial T)_{\varepsilon_y^e \varepsilon_x^e}$$

Then

$$(\partial S/\partial \varepsilon_x^p)_{T,\varepsilon_x} = V(\partial \sigma_x/\partial T)_{\varepsilon_x^e \varepsilon_y^e} - V(\partial \sigma_y/\partial T)_{\varepsilon_x^e \varepsilon_y^e}$$

$$= 2V(\partial \tau/\partial T)_{\varepsilon_x^e \varepsilon_y^e} \quad (63)$$

31

The derivative on the r.h.s. of Eq. (63) is taken with $\varepsilon_x{}^e, \varepsilon_y{}^e$ constant. Since this insures that no strain is occurring,

$$(\partial\tau/\partial T)_{\varepsilon_x{}^e \varepsilon_y{}^e} = (\partial\tau/\partial T)_{\varepsilon_x{}^e, \varepsilon_x{}^e} = (\partial\tau/\partial T)_{\varepsilon_x{}^e, \gamma} \quad \text{etc.}$$

In order to evaluate changes in entropy under the assumptions of this section, dQ of Eq. (43) must be specified. For an adiabatic process, dQ = 0, but the sample gets hotter as the process continues, so a repeating cycle doesn't exist.

To calculate dQ for an isothermal process consider incremental compression from a reference state under three different conditions, Fig. 5. Compression along AD is adiabatic and is accompanied by plastic work ΔW_p; no heat is absorbed. Compression along AC is isentropic; the amount of heat removed just compensates for the plastic work. Compression along AB is isothermal and it is along this path that we wish to calculate ΔQ. For the situation shown $\Delta Q < 0$, i.e. heat must be removed to compensate for the plastic work. By the first law the change in internal energy in going from A to B is equal to the total work done plus the heat absorbed,

$$E_B - E_A = dW + dQ \tag{64}$$

where

$$\Delta W = V\sigma_x \, d\varepsilon_x$$

According to Appendix F, dQ can be expressed in terms of incremental changes in specific volume, elastic strain and plastic work:

$$dQ = T(\partial p/\partial T)_{V,\gamma^e} \, dV - (8VT/3)(\partial\tau/\partial T)_{V,\gamma^e} \, d\gamma^e - dW_p \tag{65}$$
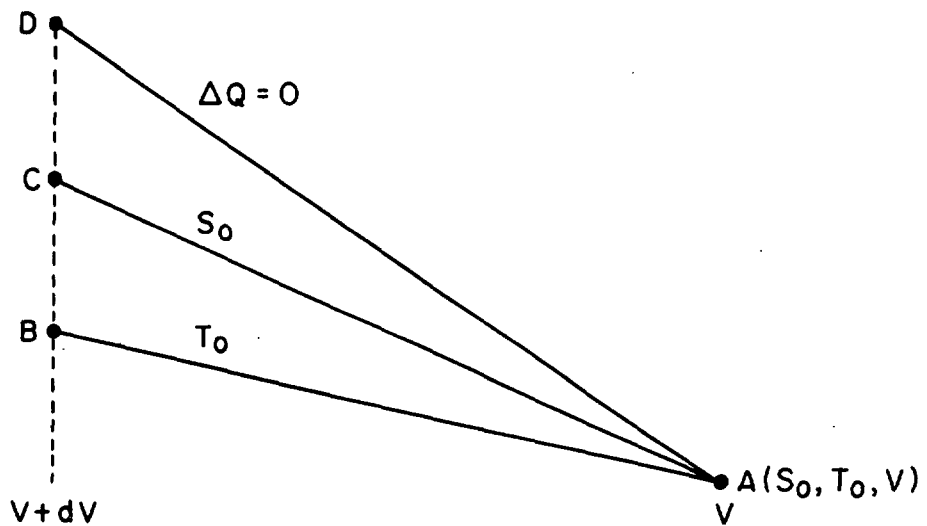
Figure 5　Heat absorbed in isothermal compression

The net entropy production in the sample is obtained by combining Eqs. (43) and (65):

$$dS = (1/T)(dW_p + dQ)$$

$$= (\partial p/\partial T)_{V,\gamma^e} \, dV - (8V/3)(\partial \tau/\partial T)_{V,\gamma^e} \, d\gamma^e \qquad (66)$$

This is a remarkable result and one which has been noted by Bridgman for uniaxial stress: Entropy absorbed by the sample under isothermal conditions is given by the elastic changes alone, even in the presence of plastic work.

Equation (66) may be obtained more directly with minimal reference to the physical processes by using the fact that entropy is a state function, so changes in entropy can be obtained by direct differentiation. Thus from Eq. (49) with $Vd\epsilon_x$ replaced by dV,

$$S = S(T, V, \gamma^e)$$

$$dS = (\partial S/\partial T)_{V,\gamma^e} \, dT + (\partial S/\partial V)_{T,\gamma^e} \, dV + (\partial S/\partial\gamma^e)_{T,V} \, d\gamma^e \qquad (67)$$

Since S is derived from the Helmholtz potential A and

$$dA = -SdT - pdV + (8V\tau/3) \, d\gamma^e,$$

the last two coefficients in Eq. (67) can be obtained by equating cross-derivatives of A:

$$\partial S/\partial V = -\partial^2 A/\partial T\partial V = \partial p/\partial T \qquad (68)$$

$$\partial S/\partial\gamma^e = -\partial^2 A/\partial T\partial\gamma^e = -(8V/3)(\partial\tau/\partial T) \qquad (69)$$

Setting $dT = 0$ and substituting Eqs. (68) and (69) into (67) yields Eq. (66). While this procedure is economical in mathematical operations, it does not reveal the physical content of the result so clearly as the earlier procedure.

Since dS is an exact differential, the cross derivatives of its coefficients must be equal. Thus

$$(\partial C_{V\gamma}/\partial V)_{T,\gamma^e} = (T\,\partial^2 p/\partial T^2)_{V,\gamma^e} \tag{70a}$$

$$(\partial C_{V\gamma}/\partial\gamma^e)_{V,T} = -(8V/3)(\partial^2\tau/\partial T^2)_{V,\gamma^e} \tag{70b}$$

$$\partial^2 p/\partial T\partial\gamma^e = -(8/3)\,\partial(V\partial\tau/\partial T)/\partial V \tag{70c}$$

The bulk modulus, K, may depend on γ^e, thus leading to a dependence of p on γ^e, but this is likely to be small. Setting it equal to zero implies that

$$\partial(V\partial\tau/\partial T)/\partial V = 0$$

or

$$V(\partial\tau/\partial T)_{V,\gamma^e} = f(\gamma^e,T) \tag{71}$$

In Eqs. (70) $C_{V\gamma}$ has been written for $(1/T)(\partial S/\partial T)_{V,\gamma^e}$. Here again the dependence of $C_{V\gamma}$ on γ^e is likely to be small so

$$(\partial^2\tau/\partial T^2)_{V,\gamma^e} \simeq 0. \tag{72}$$

Most materials will be found to yield before the linear relation between τ and γ^e is violated. Setting $\gamma = 2\mu\,\gamma^e$ with μ independent of γ^e, Eq. (71) becomes

$$(\partial\mu/\partial T)_{V,\gamma^e} = g(T)/V \tag{73}$$

while Eq. (72) suggests that

$$(\partial^2\mu/\partial T^2)_{V,\gamma^e} \simeq 0. \tag{74}$$

Equations (73) and (74) together imply that

$$\partial \mu / \partial T = b/V \tag{75a}$$

or

$$\mu = \mu_0 + b(T/V - T_0/V_0) + \phi(V) \tag{75b}$$

where $\phi(V_0) = 0$. If experiments reveal that $\partial \mu / \partial T$ depends on temperature (which it certainly does near the melting temperature) then the assumption that C_{V_γ} is independent of γ^e is invalid.

For computational purposes it is convenient to write

$$(\partial p / \partial T)_{V,\gamma^e} = \Gamma C_{V_\gamma}/V \tag{76}$$

where Γ is the Grüneisen parameter. This coefficient is slowly varying and normally, but not necessarily, positive. Taking $\tau = 2\mu\gamma^e$ as before, the second coefficient in Eq. (66) becomes

$$- (16V \gamma^e/3) (\partial \mu / \partial T)_{V,\gamma^e} \tag{77}$$

Since $\partial \mu / \partial T$ is normally negative, this coefficient has the sign of γ^e, which is frequently, but not always, negative in compression and positive in tension. Taking ΓC_V and $V \partial \mu / \partial T$ to be constants, we have for the entropy change

$$\oint dS = \Gamma C_V \oint d\varepsilon_x + (8V/3) |\partial \mu / \partial T| \oint d(\gamma^e)^2 \equiv 0 \tag{78}$$

This illustrates in a particular case the general result which follows from the assumption that E and A are state functions.

Equation (78) refers to entropy change in the sample during an isothermal cycle. The entropy of the reservoir in which the sample is imbedded will be found to increase in a cycle in the amount

$$\Delta S = \oint (dW^p/T).$$

A simple example of the calculation outlined in this section is detailed in Appendix D and in Figs. 6 and 7. In this example it is assumed that the material is linearly elastic up to the yield point (point A in Fig. 4) and that the yield stress, $Y > 0$, is constant throughout the isothermal process. Under these conditions the cycle OABFO of Fig. 4 can be followed explicitly but simply. The turning point F is determined by equating the tensile stress developed along the curve of elastic expansion, OF, to the stress along DF. With F determined in this way, stress, elastic strain, plastic strain and entropy of the sample return to their starting values at O, whereas the plastic work of deformation and the entropy of the heat reservoir do not. These results are shown in Figs. 6 and 7. With non-linear elastic behavior the computation is more difficult but results are essentially the same.

A similar calculation can be made for the isentropic case, curve AC of Fig. 5. Then elastic constants are isentropic instead of isothermal and T must be calculated in order to determine entropy delivered to the reservoir. In this case entropy of the reservoir increases monotonically. Temperature is a state function of S, V, γ^e according to Eqs. (46) and (48), as

$$dT = (\partial T/\partial S)_{V,\gamma^e} \, dS + (\partial T/\partial V)_{S,\gamma^e} \, dV + (\partial T/\partial \gamma^e)_{S,V} \, d\gamma^e \, .$$

Set $dS = 0$ for isentropic compression and use the equalities among cross-derivatives of E to obtain

$$dT = -(\partial p/\partial S)_{V,\gamma^e} \, dV + (8V/3)(\partial \tau/\partial S)_{V,\gamma^e} \, d\gamma^e \, . \qquad (79)$$

It will now be found that T goes through a maximum and a minimum during each cycle and returns to its initial value at the end of a cycle, since dT is an
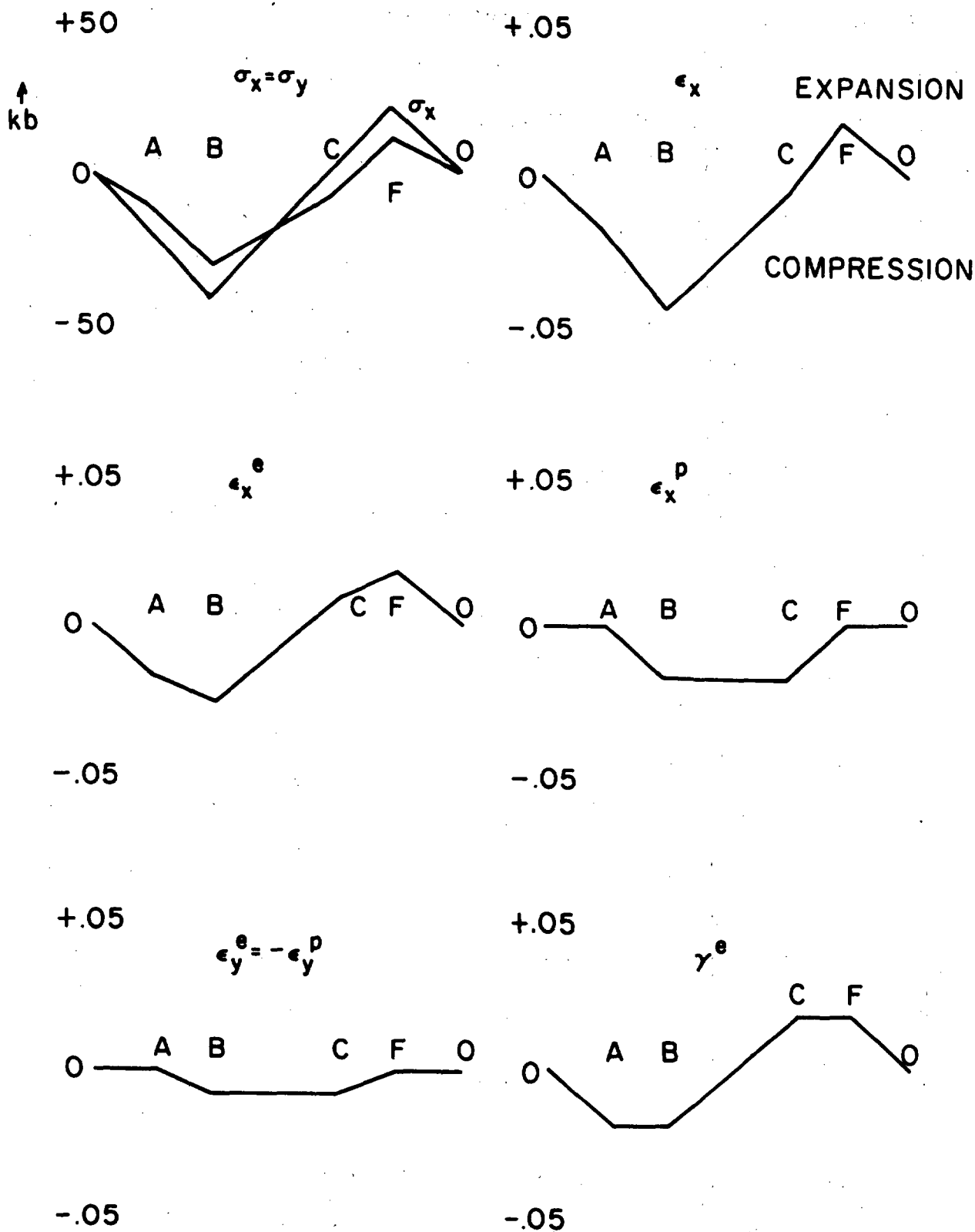
Figure 6 Strain and stress variations for isothermal cycle of linear elastic-plastic material with constant yield stress. (See Appendix D.) Abscissa is $\int |d\sigma_x|$.
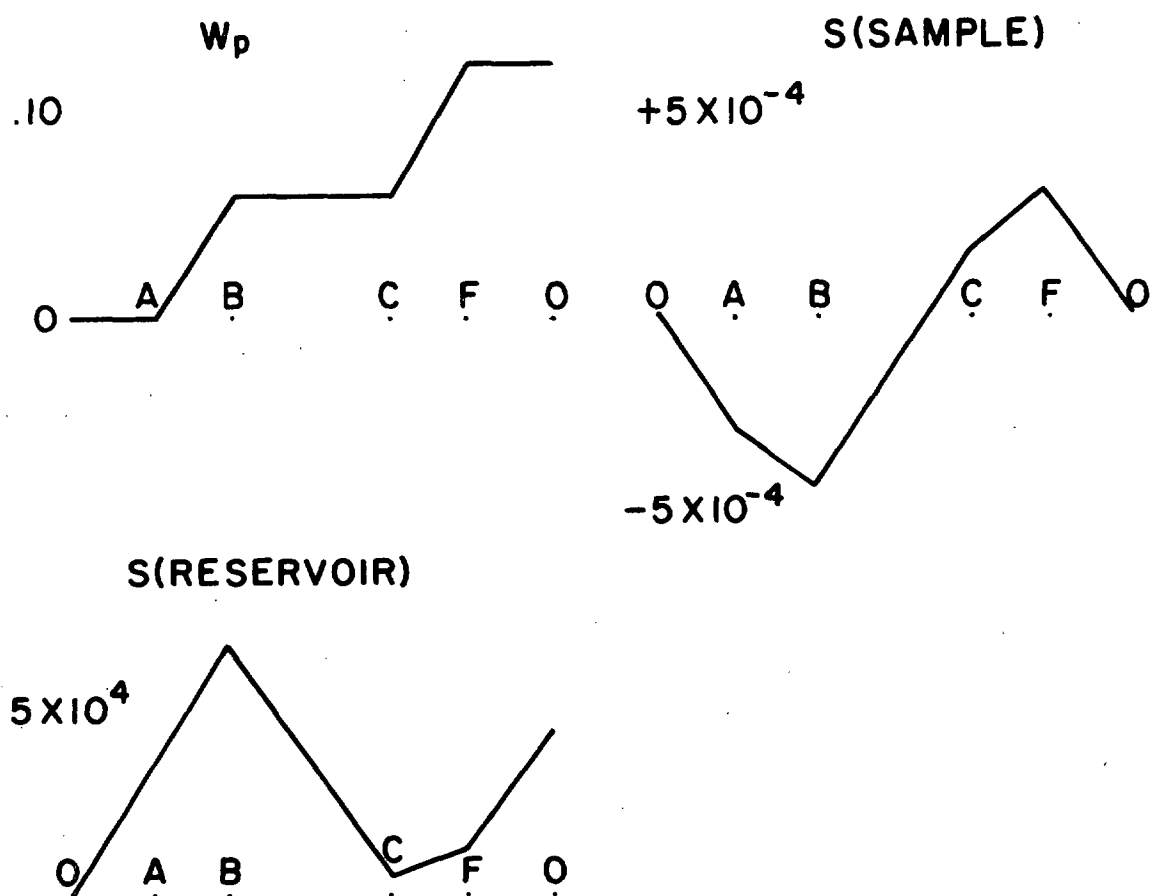
Figure 7   Entropy and plastic work for the example of Figure 6.

exact differential. Since the dependence of observables on entropy is not measured directly, this case appears to be of lesser interest than the isothermal case.

Heat conduction effects are not normally significant in stress wave propagation, so compression is inclined to be adiabatic. V, γ^e, S, T do not repeat cyclically as stress is cycled because plastic work is retained in the sample as heat. One can ask for the fate of mechanical and thermal variables as p_x increases to its maximum value and returns to zero. It may be of further interest to inquire what processes are required to return it from this final state to its initial state, but this is not normally of interest, for example, in a shock wave problem.

In the adiabatic case, both entropy and temperature are changed with each increment of plastic work. The sample is isolated from the universe so the only source of entropy is the plastic work done on the specimen. Then

$$dS = dW_p/T .$$ (80)

Combining this with Eqs. (67) to (69) yields

$$dT = (T/C_{V\gamma}) [dW_p/T - (\partial p/\partial T)_{V,\gamma^e} dV$$

$$+ (8V/3) (\partial \tau/\partial T)_{V,\gamma^e} d\gamma^e]$$ (81)

4.2 Plastic Work Partially Converted to Internal Strains. Thermodynamic Potentials

The admission that plastic work changes material properties has rather profound effects on thermodynamic concepts though the magnitude of energy involved is not very large; probably not more than 5 to 10% of the plastic work is stored in this way, and the total may be much less. The most directly

observed effect of internal energy storage is work-hardening; in some materials the yield stress may be doubled by plastic work. Although this effect results from partition of the plastic work, it is in itself not a thermodynamic effect and so must be taken as an empirical result.

In order to treat this case thermodynamically, we still consider that at each point on a deformation path the material is in a reproducible thermodynamic state, but that the state depends on an internal variable as well as the external variables of the previous section. The physical and mechanical characteristics of plastically deformed material are commonly ascribed to the creation and entanglement of dislocations, so dislocation parameters are used as internal thermodynamic variables. Dillon and Kratchovil (1970) have used dislocation density and dislocation entanglement as two internal variables. However the present state of understanding of dislocations hardly seems to justify the use of two variables. Here the dislocation density will be taken as the sole internal variable. Let this be denoted by N, the total dislocation length per unit volume of material. Denote the energy stored per unit length of dislocation by $\rho r \simeq b^2 \mu/2$ (Nabarro, p. 694). Then changes in internal energy are reversible in the variables S, V, γ^e, N:

$$dE = TdS - pdV + (8V\tau/3)\, d\gamma^e + rdN, \qquad (82)$$

implying that

$$E = E(S, V, \gamma^e, N) \qquad (83)$$

$$A = A(T, V, \gamma^e, N). \qquad (84)$$

These potentials being given, the change in entropy resulting from isothermal compression can be calculated as before (dT = 0):

$$dS = (\partial S/\partial V)_{T,\gamma^e,N}\, dV + (\partial S/\partial \gamma^e)_{T,V,N}\, d\gamma^e$$

$$+ (\partial S/\partial N)_{T,V,\gamma^e}\, d\gamma^e \,.$$

Using the compatibility relations derived from A this can be written

$$dS = (\partial p / \partial T)_{V, \gamma^e, N} \, dV - (8V/3)(\partial \tau / \partial T)_{V, \gamma^e, N} \, d\gamma^e$$

$$+ (\partial r / \partial T)_{V, \gamma^e, N} \, dN \tag{85}$$

4.3 Compatibility Relations

dS is an exact differential since it is derived from the state function, Eq. (84). This implies that the coefficients in Eq. (85) satisfy the following conditions:

$$(\partial^2 p / \partial T \partial \gamma^e)_{V, N} = -(8/3)\{\partial[V(\partial \tau / \partial T)_{V, \gamma^e, N}] / \partial V\}_{\gamma^e, N} \tag{86a}$$

$$(\partial^2 p / \partial N \partial T)_{V, \gamma^e} = -(\partial^2 r / \partial V \partial T)_{\gamma^e, N} \tag{86b}$$

$$(\partial^2 r / \partial T \partial \gamma^e)_{V, N} = (8/3)\{\partial[V(\partial \tau / \partial T)_{V, \gamma^e, N}] / \partial N\}_{V, \gamma^e} \tag{86c}$$

Assume that yield always occurs at sufficiently small shear that shear stress and strain are linearly related,

$$\tau = 2\mu\gamma^e \tag{87}$$

$$\partial \mu / \partial \gamma^e \equiv 0. \tag{88}$$

Assume also that pressure is independent of elastic shear strain. While this is not necessarily true, it is highly plausible:

$$\partial p / \partial \gamma^e \equiv 0. \tag{89}$$

From Nabarro (p. 694) one finds that

$$r \simeq b^2 V\mu/2. \tag{90}$$

42

The implications of Eqs. (86) to (90) are derived in Appendix I. They are

$$(\partial p/\partial T)_{\gamma^e,V,N} = f_2(T,V) \tag{91}$$

$$(\partial(\mu V)/\partial T)_{\gamma^e,V,N} = f_1(T) \tag{92}$$

or

$$p = f(T,V) + g(V,N) \tag{93}$$

$$\mu V = F(T) + G(V,N) \tag{94}$$

Equations (91) to (94) are useful in constructing equations of state for the classes of materials described here. For example, Eq. (93) limits the Helmholtz potential to the form

$$A(T,V,\gamma^e,N) = - \int f(T,V)dV - \int g(V,N)dV + \phi(T,N,\gamma^e) \tag{95}$$

where, f, g and ϕ are arbitrary functions of the variables indicated.

The possibility of deriving relations like those of Eqs. (91) to (95) is a consequence of the assumption of uniaxial strain, on which this entire calculation is based. Under conditions of uniaxial strain it is possible for density changes and plastic strain to occur simultaneously. In uniaxial stress this is not the case: thermodynamic potentials do not depend on both V and γ, so compatibility relations of the kind derived above do not exist. It does not seem likely that going to a more general process than uniaxial strain would produce additional information, except in anisotropic materials.

4.4   Gibbs Relation Including Stored Energy

The Gibbs relation for entropy production in an arbitrary process can be derived as before. Equate the increase in internal energy of unit mass to the sum of work done and heat added; this gives the analogue of Eq. (42):

$$TdS + dW_e + rdN = dW + dQ \tag{96}$$

and the Gibbs relation becomes

$$TdS = dW - dW_e - rdN + dQ$$

$$= dW_p - rdN + dQ . \tag{97}$$

Equation (97) is a formalization of the basic assumption that energy stored in dislocations can be recovered. Plastic work may be required to move dislocations, but this is directly associated with plastic strain and appears as heat. Since the energy of dislocations derives from local strain fields, internal strain could equally well be used as a parameter. N is more convenient since extensive work exists on the properties of dislocations and their behavior. When N is sufficiently large, dislocations interact and their energy is no longer proportional to N. Such interactions can be partially accounted for by letting r be a function of N. But we shall ignore them here and introduce self-annihilation and annealing of dislocations to prevent N from growing indefinitely.

If the compression process is carried out at a finite rate, Eq. (97) gives the rate of entropy production:

$$TdS/dt = dW_p/dt - rdN/dt + dQ/dt \tag{98}$$

In the event that either dW_p/dt or dN/dt is rate-dependent, this will lead to entropy production beyond that given by Eq. (85).

4.5   Equilibrium Values of N; An Anomaly

In order to complete the thermodynamic description, a procedure is required for calculating N. Two routes are available for obtaining the required information. One is to continue the formal thermodynamic description of the process and so to determine the bounds on equilibrium values set by the

compatibility relations. Such information can be augmented by statistical calculations for determination of entropy and lattice calculations for determining such parameters as r. The latter has already been noted in the preceding paragraph. The second is to apply some of the formalism of dislocation theory and expressions for multiplication rates to relate, for example, dislocation density to strain history. Both of these routes are considered here. We start with the thermodynamic development.

The great question about the thermodynamics of dislocations is whether equilibrium ever exists in the sense that the appropriate potential is minimized with respect to the various parameters. Here we assume that an equilibrium density N exists and we find how it is related to other material parameters. For this purpose we consider Eq. (82) and its companion for dA:

$$dE = TdS - pdV + (8V/3)\, d\gamma^e + rdN \tag{82}$$

$$dA = -SdT - pdV + (8V/3)\, d\gamma^e + rdN. \tag{99}$$

Because Eqs. (82) and (99) are exact differentials, the cross derivatives of their coefficients must be equal. Those involving N are

$$(\partial T/\partial N)_{S,V,\gamma^e} = (\partial r/\partial S)_{V,\gamma^e,N} \tag{100a}$$

$$(\partial p/\partial N)_{S,V,\gamma^e} = -(\partial r/\partial V)_{S,\gamma^e,N} \tag{100b}$$

$$(16V\gamma^e/3)\,(\partial \mu/\partial N)_{S,V,\gamma^e} = (\partial r/\partial \gamma^e)_{S,V,N} \tag{100c}$$

$$(\partial S/\partial N)_{T,V,\gamma^e} = -(\partial r/\partial T)_{V,\gamma^e,N}$$

$$= (b^2V/2)\,(\partial \mu/\partial T)_{V,\gamma^e,N} \tag{100d}$$

$$-(\partial p/\partial N)_{T,V,\gamma^e} = (\partial r/\partial V)_{T,\gamma^e,N} \qquad (100e)$$

$$(16V\gamma^e/3)(\partial\mu/\partial N)_{T,V,\gamma^e} = (\partial r/\partial\gamma^e)_{T,V,N} \qquad (100f)$$

Here the approximations of Eqs. (87), (88) and (90) have been used.

To see the utility of Eqs. (100), consider Eq. (100a). The right hand side can be written

$$(\partial r/\partial S) = (b^2 V/2)(\partial\mu/\partial S)_{V,\gamma^e,N}$$

$$= (b^2 V/2)(\partial\mu/\partial T)_{V,\gamma^e,N}(\partial T/\partial S)_{V,\gamma^e,N}$$

$$= (b^2 VT/2C_{V\gamma N})(\partial\mu/\partial T)_{V,\gamma^e,N} \qquad (101a)$$

where

$$C_{V\gamma N} \equiv T(\partial S/\partial T)_{V,\gamma^e,N} \; . \qquad (101b)$$

By regarding N as a function of (T, V, γ^e, p), we have

$$(\partial N/\partial T)_{S,V,\gamma^e} = (\partial N/\partial T)_{V,\gamma^e,p} + (\partial N/\partial p)_{T,V,\gamma^e}(\partial p/\partial T)_{S,V,\gamma^e} \; .$$

It seems quite plausible that p is insensitive to N; then the second term is negligible and

$$(\partial N/\partial T)_{S,V,\gamma^e} \simeq (\partial N/\partial T)_{V,\gamma^e,p} \qquad (101c)$$

Equations (100a), (101a) and (101c) combine to give

$$(\partial N/\partial T)_{V,\gamma^e,p} = 2C_{V\gamma N}/[b^2 VT(\partial\mu/\partial T)_{V,\gamma^e,N}] \qquad (102)$$

This is a curious and unexpected result. $\partial\mu/\partial T$ is known to be negative and $C_{V_\gamma N}$ is positive, therefore $\partial N/\partial T < 0$. But one expects the equilibrium number of dislocations to increase with T for the following reasons. Energy is required to form dislocations, so that an assembly which includes dislocations contains eigenstates of higher energy than a perfect lattice. Under equilibrium conditions these higher energy states are unoccupied at low temperatures but at sufficiently high temperatures they may be occupied. Just where the argument leading to Eq. (102) fails is not clear, but it is clear that the reason for the failure must be determined before other relations obtained from the compatibility condition can be safely used.

Changes in $C_{V_\gamma N}$ with N, γ^e and V can be obtained from the expression for dS when $dT \neq 0$. Equation (85) is then replaced by

$$dS = (C_{V_\gamma N}/T)dT + (\partial p/\partial T)dV - (8V/3)(\partial\tau/\partial T)\,d\gamma^e - (\partial r/\partial T)dN. \qquad (103)$$

Then

$$\partial(C_{V_\gamma N}/T)/\partial N = -\partial^2 r/\partial T^2 \qquad (104a)$$

$$\partial(C_{V_\gamma N}/T)/\partial\gamma^e = -(8V/3)\,\partial^2\tau/\partial T^2 \qquad (104b)$$

$$\partial(C_{V_\gamma N}/T)/\partial V = \partial^2 p/\partial T^2 \qquad (104c)$$

If Eqs. (87) and (90) are assumed true, Eqs. (104a) and (104b) combine to yield the result

$$\partial C_{V_\gamma N}/\partial N - (3b^2/18)\,\partial C_{V_\gamma N}/\partial(\gamma^e)^2 = 0 . \qquad (105)$$

This has solutions of the form

$$C_{V_\gamma N} = f[(N + 6\gamma^{e^2}/b^2),\ V, T], \qquad (106)$$

47

which shows an interesting relation in the dependence of $C_{V\gamma N}$ on N and γ^e. It does not appear to be susceptible to test with present information.

The physical consequences of relations like those of Eqs. (100) and (104) have been illustrated but not exhausted. For the present, however, we forsake further development of these relations.

V. MECHANICAL CALCULATIONS OF DISLOCATION DENSITY, N

5.1 Formalism

As a consequence of plastic work, dislocation densities may be generated which are much greater than equilibrium values calculated in the previous section. These may represent substantial quantities of stored energy, and at sufficiently high temperatures this should be released as heat with a consequent reduction of N. The thermodynamic effects of non-equilibrium changes of N are derived from the Gibbs relation in the form of Eq. (100). This indicates that dN/dt is the quantity of prime interest.

A form frequently used for dN/dt is quoted by Webster (1966):

$$dN/dt = aN - bN^2 + S_0. \qquad (107)$$

The first term represents creation of dislocations by regenerative processes, the second represents self-annihilation, and the third represents spontaneous nucleation of dislocations under the influence of externally applied forces. Equation (107) can be integrated directly to yield

$$N = [A - B\theta \exp(-\kappa t)] / [1 + \theta \exp(-\kappa t)] \qquad (108)$$

where $\kappa = (a^2 + 4b \, S_0)^{1/2}$

$\quad A = (a/2b) (1 + \kappa/a)$

$\quad B = -(a/2b) (1 - \kappa/a)$

$\quad \theta = (A - N_i)/(B + N_i)$

$\quad N_i = $ dislocation density at t = 0

For $S_0 \equiv 0$,

$$N = a \, N_i \exp(at) / [a + N_i b (\exp (at) - 1)]. \qquad (109)$$

49

The Gilman relation for multiple cross glide has

$$aN = mbv \quad N = N_i \, f \, \dot{\gamma}^p$$

where $\dot{\gamma}^p$ is plastic strain rate. Then, for small multiplication, changes in N are proportional to plastic strain. This relation has been widely used in discussion of dynamic failure (Asay, et al. 1972).

For large t and $S_0 = 0$, Eq. (109) goes to

$$N \equiv N_S = a/b \tag{110}$$

This is the saturation value of N; it is unrelated to the thermal equilibrium value discussed in the previous section. From Eq. (109), $1/a \equiv t_e$ is a characteristic time for equilibration. Then Eq. (110) can be rewritten in the form

$$1/N - 1/N_S = (1/N_i - 1/N_S) \exp(-t/t_e); \tag{111}$$

t_e is assumed to be the order of a few multiplication times and is discussed in Appendix H:

$$t_e = \alpha \delta / v_\delta - (\alpha/n)/ \ln [1 - \exp(-U/kT)] \tag{112}$$

where δ = distance between imperfections which cause multiplication

n = a characteristic frequency associated with pinned dislocations

$U = U_0 (1 - \sigma/\sigma_0)$ is an activation energy

v_δ = shear wave velocity

α = a number the order of ten

σ = effective stress driving the dislocation

σ_0 = critical or threshold stress

Equation (112) is to be applied if $\sigma < \sigma_0$. For $\sigma \geq \sigma_0$,

$$t_e = \alpha \delta / v_\delta. \tag{113}$$

If $U/kT \sim 10$, Eq. (112) gives $t_e \sim 10^{-4}$ seconds, which is very long on the scale of shock experiments. On the other hand, Eq. (113) is the order of 10^{-12} for ~ 100 pinning points per million lattice sites, a time which is completely negligible.

5.2 Applications to Shock Processes

If we assume the Gilman relation for multiplication and ignore self-annihilation, we have

$$dN/dt = mbvN = m\dot{\gamma}^p$$

or

$$dN = md\dot{\gamma}^p.$$

Each increment in plastic work is also proportional to $d\gamma^p$:

$$dW_p = (8V\tau/3)d\gamma^p.$$

The increment in stored energy is rdN and its ratio to dW_p is

$$rdN/dW_p = 3mr/8V\tau = 3b^2\mu m/16\tau .$$

For $m = 10^{11}$, $b = 3 \times 10^{-8}$, $\mu = 2.7 \times 10^{11}$ and $\tau = 5 \times 10^9$ dynes/cm^2, this ratio is

$$rdN/dW_p \approx .9 \times 10^{-3} = .09\%$$

This value is to be compared with the oft-quoted value of "5 to 10%." This can be achieved by making τ very much smaller, as in a static test, or by increasing m. A value of 0.5 kilobars in a static test on soft aluminum is not out of reason, in which case the ratio of stored energy to plastic work goes to 0.9%. Moreover m is not precisely known and may be increased several-fold without seriously violating any experiments. In fact some of the dynamic failure data

51

for aluminum suggest that m may be as large as 2.5×10^{12} in shock experiments. Then taking $\tau = 5$ kb again,

$$rdN/dW_p \simeq 2.3\%$$

which is approaching the usually quoted range for stored energy. The possibility that plastic stored energy in shock compression is less than static values must be admitted.

For comparison, a similar calculation for the case of uniaxial stress leads to the ratio

$$rdN/dW_p = mr/VY = mb^2\mu/2Y$$

where $Y = 2\tau$ is the yield stress in simple tension. This does not differ significantly from the value for uniaxial strain.

The total plastic work generated when aluminum is shocked to 100kb is approximately $W_p = .15$ kb cc/g or 1.5×10^8 ergs/g. Equating 5% of this to rN where $r \simeq 4.5 \times 10^{-5}$ gives

$$N = .05 \times 1.5 \times 10^8/4.5 \times 10^{-5} = 1.6 \times 10^{11}.$$

This is small for a saturation value of N and it suggests that in the small times available under shock conditions, saturation of dislocation density may not occur.

VI. DISCUSSION AND CONCLUSIONS

A simple and approximate model for treating the thermodynamics of elastic-plastic solids has been initiated here but not completed. Its application to conditions of unixial strain shows that the additional thermodynamic degree of freedom compared to uniaxial stress gives some potentially interesting new compatibility relations which can be checked against various experimental data. It is also suggested that shock experiments which have uniaxial strain geometry may be useful in evaluating dislocation concepts and those of elastic-plastic thermodynamics.

The point of view espoused here is a novel one. It is common to speak of an elastic-plastic system as being "essentially irreversible" or "immersed in a sea of irreversibility." The view expressed here, but not completely developed, is that the elastic-plastic state is completely reversible in the variables which are natural to the state and that the apparent irreversibility associated with inability to retrace a α-ϵ state is an awkward consequence of geometry and is not essentially related to the question of reversibility of the process. Having stated such a hypothesis, one must then make some decision about the role of stored plastic work. For many purposes the problem is indifferent to this decision because the quantity of stored energy is small. But when the stored energy is related to dislocation density, the thermodynamic formalism suggests that this may provide some new avenues for investigation of the role of dislocations in plastic deformation.

APPENDICES

APPENDIX A

Addition of Elastic and Plastic Strain Increments

In Fig. Al are shown two points of a continuum, O and P, separated by a small distance $\vec{s}$. Application of an external stress forces O to A and P to B. When the stress is removed, A relaxes to Q and B to T. We have from the figure

$$\vec{OA} = \vec{u}, \quad \vec{PB} = \vec{u} + d\vec{u}$$
$$d\vec{u} = \vec{s} \cdot \nabla\vec{u} \tag{A1}$$
$$\vec{AQ} = \vec{u}', \quad \vec{BT} = \vec{u}' + d\vec{u}'$$
$$d\vec{u}' = (\vec{s} + d\vec{u}) \cdot \nabla\vec{u}' \tag{A2}$$
$$\vec{OQ} = \vec{u}'', \quad \vec{PT} = \vec{u}'' + d\vec{u}''$$
$$d\vec{u}'' = \vec{s} \cdot \nabla\vec{u}'' \tag{A3}$$

In writing Eqs. (A1) - (A3) the vector change $d\vec{u}$, etc has been expanded in a Taylor series in $\vec{s}$ or $\vec{s} + d\vec{u}$ and all but the lowest order term discarded.

$d\vec{u}''$ is identified as the plastic displacement, $-d\vec{u}'$ the elastic displacement and $d\vec{u}$ as the total displacement. Then

$$-d\vec{u}' + d\vec{u}'' = \vec{s} \cdot (-\nabla\vec{u}' + \nabla\vec{u}'') - d\vec{u} \cdot \nabla\vec{u}'$$
$$= \vec{s} \cdot (-\nabla\vec{u}' + \nabla\vec{u}'' - \nabla\vec{u} \cdot \nabla\vec{u}') \tag{A4}$$

The vector gradient, $\nabla\vec{u}$, can be decomposed into a strain tensor $\underline{\varepsilon}$ and a rotation tensor $\underline{\theta}$:

$$\nabla\vec{u} = \underline{\varepsilon} + \underline{\theta}$$
$$\text{where } \underline{\varepsilon} = (1/2)(\nabla\vec{u} + \widetilde{\nabla\vec{u}})$$
$$\underline{\theta} = (1/2)(\nabla\vec{u} - \widetilde{\nabla\vec{u}})$$

and $\widetilde{\nabla\vec{u}}$ is the transpose of $\nabla\vec{u}$.

Then Eq. (A4) becomes

$$-d\vec{u}' + d\vec{u}'' = \vec{s} \cdot (-\underline{\varepsilon}' + \underline{\varepsilon}'' - \underline{\varepsilon}' \cdot \underline{\varepsilon}'' - \underline{\theta}' \cdot \underline{\theta}'' + \text{rotation terms}).$$

Then if $\underline{\varepsilon}'$, $\underline{\varepsilon}''$, $\underline{\theta}'$, $\underline{\theta}''$ are infinitesimal, the 2nd order terms can be neglected so

that

$$d\vec{u} = -d\vec{u}' + d\vec{u}'' = \vec{s} \cdot (-\underline{\underline{\varepsilon}}' + \underline{\underline{\varepsilon}}'' + \text{rotation terms}) = \vec{s} \cdot (\underline{\underline{\varepsilon}} + \underline{\underline{\theta}})$$

so that

$$\underline{\underline{\varepsilon}} = -\underline{\underline{\varepsilon}}' + \underline{\underline{\varepsilon}}'' = \underline{\underline{\varepsilon}}^e + \underline{\underline{\varepsilon}}^p \qquad (A5)$$

where $\underline{\underline{\varepsilon}}^e$ has been set equal to $-\underline{\underline{\varepsilon}}'$.   In a finite deformation process the relation can be applied to each increment in strain so that

$$d\underline{\underline{\varepsilon}} = d\underline{\underline{\varepsilon}}^e + d\underline{\underline{\varepsilon}}^p \qquad (A6)$$
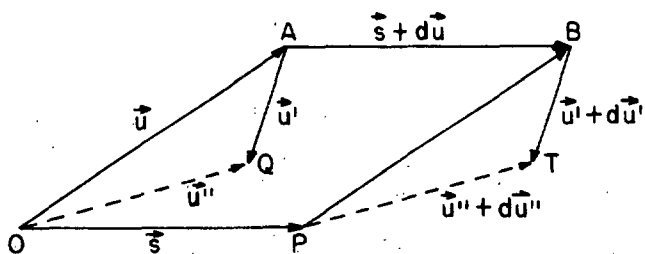
Figure A1   Arbitrary displacements in
a continuum

58

APPENDIX B

Stresses in Elastic-Plastic Deformation

The conventional assumption in elastic-plastic theory is that stress is supported by the elastic strain and there exists no contribution from the plastic strain. This appears plausible if one examines the microscopic behavior of a plastically deforming material from the point of view of dislocation theory.

Plastic deformation is synonymous with motion and generation of dislocations. In most materials dislocations probably move at all stress levels, so there is no such thing as a yield point and there is no truly elastic behavior. But in practice the yield point concept is useful and the yield point itself may be taken to be the point at which large numbers of dislocations are set in motion. The motion of dislocations is inhibited by the existence of energy barriers which must be overcome by combination of stored elastic energy and thermal fluctuations of atoms. So, for example, one may consider a strained, work hardening solid as one in which many dislocations exist but are momentarily immobilized or "pinned" by energy barriers. If applied stress is held constant, dislocations may occasionally overcome a barrier and move to the next barrier, thus contributing to the plastic deformation by creep.

If the external stress is increased, local strain energy is increased around the pinning points, more dislocations are moved through pinning points and pass on until they are pinned again. If it is assumed that dislocations move freely between pinning points, the plastic strain which results from their motion requires no part of the applied stress; i.e. the applied stress is supported entirely by the elastic part of the strain. It does follow, however, that a relaxation process may exist. For example, an increment in stress increases local strain energy around pinning points and increases probability

59

that dislocations will break free of their pinning points and move on until pinned again. However, the probability of a dislocation breaking free is time dependent because it depends on thermal fluctuations as well as strain energy. This means that an increment in stress will produce an immediate elastic strain, and that, as time passes, dislocations escape their pins, move, and are repinned, so that a plastic strain develops. Or the original increment in strain is reduced as the pinned dislocations escape from their barriers and the elastic strain is reduced. This concept can be expressed quantitatively in the following way

$$d\sigma = \rho a^2 d\epsilon^e \tag{B1}$$
$$= \rho a^2 (d\epsilon - d\epsilon^p)$$
$$= \rho a^2 d\epsilon - a^2 (d\epsilon^p/dt)dt$$

or
$$d\sigma/dt = \rho a^2 d\epsilon/dt - \rho a^2 d\epsilon^p/dt.$$

In terms of stress and strain deviators this translates to

$$dS_j/dt = 2\mu\, de_j/dt - 2\mu de_j^p/dt$$
$$= 2\mu\, de_j/dt - F(S,e) \tag{B2}$$

where F is a relaxation function.

Equations (B1) and (B2) are both based on the assumption that dislocations move between pinning points without drag. If this is not true, then a viscous contribution to the stress appears:

$$d\sigma = \rho a^2 d\epsilon^p + \eta\, d\dot{\epsilon}_p$$

or
$$dS_j/dt = 2\mu\, de_j/dt - 2\mu\, de_j^p/dt + 2\eta\, d\dot{e}_j^p/dt \tag{B3}$$

where $\dot{e}_j^p \equiv de_j^p/dt.$

APPENDIX C

Plastic Work and Plastic Strain

From Eq. (24b)

$$d\epsilon_x^p = d\epsilon_x - d\epsilon_x^e \tag{C1}$$

Eq. (24a) $\quad d\theta = d\epsilon_x + 2d\epsilon_y = d\epsilon_x^e + 2d\epsilon_y^e + (d\epsilon_x^p + 2d\epsilon_y^p)$

$$= d\epsilon_x^e + 2d\epsilon_y^e + 0$$

$$= d\epsilon_x \quad \text{by Eq. (28a).}$$

So

$$d\epsilon_y^e = d\epsilon_x/2 - d\epsilon_x^e/2 = d\epsilon_x^p/2$$

$$d\sigma_x = \lambda d\theta + 2\mu\, d\epsilon_x^e \quad \text{by Eq. (27)}$$

$$d\sigma_y = \lambda d\theta + 2\mu\, d\epsilon_y^e \quad \text{by Eq. (27)}$$

$$d(\sigma_x - \sigma_y) = 2\mu(d\epsilon_x^e - d\epsilon_y^e) = 2\mu(d\epsilon_x^e - d\epsilon_x^p/2) \tag{C2}$$

$$d\tau = [d(\sigma_x - \sigma_y)]/2 = \mu\, d\epsilon_x^e - \mu\, d\epsilon_x^p/2$$

$$d\epsilon_x^e = d\tau/\mu + d\epsilon_x^p/2 \tag{C3}$$

so $\quad d\epsilon_x^p = d\epsilon_x - d\tau/\mu - d\epsilon_x^p/2 \quad$ from Eqs. (C1) and (C3)

$$d\epsilon_x^p = (2/3)(d\epsilon_x - d\tau/\mu) \tag{C4}$$

Combine this with Eq. (36) to obtain

$$dW_p = \frac{4}{3} V\tau \cdot \frac{3}{2} d\epsilon_x^p = \underline{\underline{2V\tau d\epsilon_x^p}} \tag{C5}$$

From Eq. (25) $dW_p = V(S_x de_x^p + 2S_y de_y^p)$

$$= \frac{4}{3} V\tau (de_x^p - de_y^p) \text{ from Eq. (30) and (31)} \qquad (C6)$$

$$de_x = d\epsilon_x - d\theta/3 = d\epsilon_x^e + d\epsilon_x^p - d\theta/3$$

$$de_y = d\epsilon_y - d\theta/3 = d\epsilon_y^e + d\epsilon_y^p - d\theta/3$$

$$de_x - de_y = d\epsilon_x - d\epsilon_y$$

$$(de_x^e - de_y^e) + (de_x^p - de_y^p) = (d\epsilon_x^e - d\epsilon_y^e) + (d\epsilon_x^p - d\epsilon_y^p) \qquad (C7)$$

$$de_x^e - de_y^e = d\tau/\mu \text{ by Eq. (32)}$$

$$d\epsilon_x^e - d\epsilon_y^e = d\tau/\mu \text{ by Eq. (C2)} \qquad \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} (C8)$$

Therefore Eq. (C7) becomes

$$de_x^p - de_y^p = d\epsilon_x^p - d\epsilon_y^p$$

$$= \frac{3}{2} d\epsilon_x^p$$

since $d\epsilon_x^p + 2d\epsilon_y^p = 0$.

Substituting this into Eq. (C6) yields

$$dW_p = \frac{4}{3} V\tau \cdot \frac{3}{2} d\epsilon_x^p = 2V\tau d\epsilon_x^p$$

in agreement with Eq. (C5).

Note that the following argument is faulty:

$$de_x = d\epsilon_x - d\theta/3 = \frac{2}{3} d\epsilon_x$$

$$de_y = d\epsilon_y - d\theta/3 = \frac{1}{3} d\epsilon_x$$

62

$$de_x - de_y = d\varepsilon_x$$

$$(de_x{}^e - de_y{}^e) + (de_x{}^p - de_y{}^p) = d\varepsilon_x{}^p + d\varepsilon_x{}^e$$

$$\therefore \quad de_x{}^p - de_y{}^p = d\varepsilon_x{}^p$$

$$\text{and} \quad de_x{}^e - de_y{}^e = d\varepsilon_x{}^e$$

This result contradicts Eq. (C8) and is not true because the non-zero values of $d\varepsilon_y{}^e$ and $d\varepsilon_y{}^p$ have been ignored in obtaining the final relation.

Stress Cycles for a Linear Elastic-Plastic Material with Constant Yield Stress. All Plastic Work is Converted to Heat.

$T = 300°K$

$\lambda = $ constant $= 582$ kb

$\mu = $ constant $= 274$ kb

$Y = $ constant $= 10$ kb

$\varepsilon_y^e = \varepsilon_z^e$

$V_0 = .37$ cm^3/g

$C_V = 8.4 \times 10^{-3}$ kbcc/g°

$\Gamma = 2$

$\sigma_y = \sigma_z$

$\varepsilon_y = \varepsilon_z = 0$

$\varepsilon_y^p = \varepsilon_z^p$

$-V(\partial\sigma_x/\partial T)_{\varepsilon_x \varepsilon_x^p} = 16.8 \times 10^{-3}$ kbcc/g°

Initial Conditions at 0 (Figs. 3 and 4)

$\sigma_x = \sigma_y = 0;\quad \varepsilon_x = \varepsilon_x^e = \varepsilon_x^p = \varepsilon_y^e = \varepsilon_y^p = 0$

$W_p = 0,\quad S = 0;\quad$ entropy of heat reservoir, $S_R = 0$

Along OA (Figs. 3 & 4)

$d\varepsilon_x = d\varepsilon_x^e = d\sigma_x/(\lambda + 2\mu)$

$d\sigma_y = \lambda d\sigma_x/(\lambda + 2\mu)$

$d\varepsilon_x^p = d\varepsilon_y^p = dW_p = d\varepsilon_y^e = 0$

$dW_p = 0$

dS (reservoir) = -dS (sample)

Along AB

$\sigma_x - \sigma_y = -Y;\quad d(\sigma_x - \sigma_y) = -dY = 2\mu(d\varepsilon_x^e - d\varepsilon_y^e) = 0$

$d\varepsilon_y^e = d\varepsilon_x^e$

$$d\varepsilon_x = d\varepsilon_x{}^e + 2d\varepsilon_y{}^e = 3d\varepsilon_x{}^e$$

$$d\sigma_x = 3\lambda d\varepsilon_x{}^e + 2\mu d\varepsilon_x{}^e$$

$$d\varepsilon_x{}^e = d\sigma_x/(3\lambda + 2\mu)$$

$$d\varepsilon_x = 3d\varepsilon_x{}^e = 3d\sigma_x/(3\lambda + 2\mu)$$

$$d\varepsilon_x{}^p = d\varepsilon_x - d\varepsilon_x{}^e = 2d\varepsilon_x{}^e$$

$$dW_p = 2V\tau d\varepsilon_x{}^p = -VY d\varepsilon_x{}^p$$

$$V = V_o \exp(\varepsilon_x)$$

$$d\sigma_y = d\sigma_x$$

$$d\varepsilon_y{}^p = -d\varepsilon_y{}^e = -d\varepsilon_x{}^e$$

$$dS \text{ (reservoir)} = -dS \text{ (sample)} + dW_p/T$$

Along BC

$$d\varepsilon_x = d\varepsilon_x{}^e = d\sigma_x/(\lambda + 2\mu)$$

$$d\sigma_y = \lambda d\sigma_x/(\lambda + 2\mu)$$

$$d\varepsilon_x{}^p = d\varepsilon_y{}^p = d\varepsilon_y{}^e = 0$$

$$dW_p = 0$$

$$dS \text{ (reservoir)} = -dS \text{ (sample)}$$

66

Aong CF

$$\sigma_x - \sigma_y = Y$$

$$d\epsilon_y^e = d\epsilon_x^e$$

$$d\epsilon_x = 3d\epsilon_x^e = 3d\sigma_x/(3\lambda + 2\mu)$$

$$d\epsilon_x^p = 2d\epsilon_x/3$$

$$d\epsilon_y^p = -d\sigma_x/(3\lambda + 2\mu)$$

$$d\sigma_y = d\sigma_x$$

$$dW_p = VY\, d\epsilon_x^p$$

$$dS\ (reservoir) = -dS\ (sample) + dW_p/T$$

Along FO

$$d\epsilon_x = d\epsilon_x^e = d\sigma_x/(\lambda + 2\mu)$$

$$d\sigma_y = \lambda d\sigma_x/(\lambda + 2\mu)$$

$$d\epsilon_x^p = d\epsilon_y^p = d\epsilon_y^e = 0$$

$$dW_p = 0$$

$$dS\ (reservoir) = -dS\ (sample)$$

At 0

$$\epsilon_x = 0, \quad \sigma_x = 0$$

67

Compressive stress at B: assume $\sigma_x{}^B$ = __-40.6 kb__

$\underline{\sigma_x, \ \sigma_y}$

at A: $\sigma_x - \sigma_y = [1 - \lambda/(\lambda + 2\mu)]\sigma_x = -Y$

$\qquad \sigma_x{}^A = -(\lambda + 2\mu)Y/2\mu = \underline{-20.6 \ kb}; \quad \underline{\sigma_y{}^A = -10.6 \ kb}$

at B: $\sigma_x{}^B = \underline{-40.6 \ kb}$ (assumed), $\underline{\sigma_y{}^B = -30.6 \ kb}$

at C: $\sigma_x{}^C - \sigma_y{}^C = Y$

$\qquad \sigma_x{}^C - \sigma_y{}^C - (\sigma_x{}^B - \sigma_y{}^B) = (\sigma_x{}^C - \sigma_x{}^B) - (\sigma_y{}^C - \sigma_y{}^B)$

$\qquad\qquad = (\sigma_x{}^C - \sigma_x{}^B)[1 - \lambda/(\lambda + 2\mu)] = 2Y$

$\qquad \sigma_x{}^C - \sigma_x{}^B = Y(\lambda + 2\mu)/\mu = 41.2 \ kb$

$\qquad \underline{\sigma_x{}^C = 0.6 \ kb}$

$\qquad \underline{\sigma_y{}^C = -9.4 \ kb}$

at F: $\sigma_x{}^F - \sigma_y{}^F = Y$ whether the point F is reached by expanding along the

elastic isotherm, OF, or along the plastic curve, CF. In the former case,

$\underline{\sigma_x{}^F = -\sigma_x{}^A = 20.6 \ kb.}$ Then $\underline{\sigma_y{}^F = 10.6 \ kb}$

at O: Both σ_x and σ_y return to zero since both have their elastic values at F.

$\underline{\varepsilon_x}$

$\varepsilon_x{}^A = \sigma_x{}^A/(\lambda + 2\mu) = \underline{-.01823}$

$$\varepsilon_x^B = \varepsilon_x^A + 3(\sigma_x^B - \sigma_x^A)/(3\lambda + 2\mu) = -.01823 - .02616 = \underline{-.04439}$$

$$\varepsilon_x^C = \varepsilon_x^B + (\sigma_x^C - \sigma_x^B)/(\lambda + 2\mu) = -.04437 + .03646 = \underline{-.00793}$$

$$\varepsilon_x^F = \varepsilon_x^C + 3(\sigma_x^F - \sigma_x^C)/(3\lambda + 2\mu) = -.00793 + .026155 = \underline{+.01823}$$

$$\varepsilon_x^0 = 0$$

$\underline{\varepsilon_x^e}$

$$\varepsilon_x^e(A) = \varepsilon_x^A = -.01823$$

$$\varepsilon_x^e(B) = \varepsilon_x^e(A) + (\sigma_x^B - \sigma_x^A)/(3\lambda + 2\mu) = -.01823 - .00871 = \underline{-.02694}$$

$$\varepsilon_x^e(C) = \varepsilon_x^e(B) + (\sigma_x^C - \sigma_x^B)/(\lambda + 2\mu) = -.02694 + .03646 = \underline{.00952}$$

$$\varepsilon_x^e(F) = \varepsilon_x^e(C) + (\sigma_x^F - \sigma_x^C)/(3\lambda + 2\mu) = .00952 + .00871 = \underline{.01823}$$

$$\varepsilon_x^e(0) = 0$$

$\underline{\varepsilon_x^p}$

$$\varepsilon_x^p(A) = \underline{0}$$

$$\varepsilon_x^p(B) = 2(\sigma_x^B - \sigma_x^A)/(3\lambda + 2\mu) = \underline{-.017428}$$

$$\varepsilon_x^p(C) = \varepsilon_x^p(B)$$

$$\varepsilon_x^p(F) = \varepsilon_x^p(C) + 2(\sigma_x^F - \sigma_x^C)/(3\lambda + 2\mu) = \underline{0} = \varepsilon_x^p(0)$$

$\underline{\varepsilon_y^e}$

$$\varepsilon_y^e(A) = \underline{0}$$

$$\varepsilon_y^e(B) = (\sigma_x^B - \sigma_x^A)/(3\lambda + 2\mu) = \underline{-.008714}$$

69

$$\varepsilon_y^e(C) = \varepsilon_y^e(B)$$

$$\varepsilon_y^e(F) = \varepsilon_y^e(C) + (\sigma_x^F - \sigma_x^C)/(3\lambda + 2\mu) = \underline{0} = \varepsilon_y^e(0)$$

$$\underline{\underline{\varepsilon_y^p = -\varepsilon_y^e}}$$

$$\underline{\underline{\gamma^e = (\varepsilon_x^e - \varepsilon_y^e)/2}}$$

$$\gamma^e(A) = -.01823/2 = -.00911$$

$$\gamma^e(B) = -.00911$$

$$\gamma^e(C) = +.00911$$

$$\gamma^e(F) = .00911$$

$$\gamma^e(0) = 0.0$$

$$\underline{\underline{W_p}}$$

$$W_p(A) = 0.0$$

$$W_p(B) = -(2YV_0/3) [\exp(\varepsilon_x^B) - \exp(\varepsilon_x^A)] = \underline{.0625}$$

$$W_p(C) = W_p(B)$$

$$W_p(F) = W_p(B) + (2YV_0/3) [\exp(\varepsilon_x^F) - \exp(\varepsilon_x^C)] = .0625 + .0648 = \underline{.1273}$$

Computation of Entropy

From Eqs. (66) - (68) changes in entropy of the sample are given by

$$dS = \Gamma C_V \, d\varepsilon_x - (16V/3) (\partial\mu/\partial\Gamma)_{\gamma^e, V} \, \gamma^e \, d\gamma^e \ .$$

From Eq. (35), $d\gamma^e = d\tau/2\mu$.

Since τ is assumed constant during plastic flow, $d\gamma^e = 0$ on AB and CF. Also, $d\gamma^e = (d\varepsilon_x^e - d\varepsilon_y^e)/2$. In the absence of plastic flow, $d\varepsilon_x^e = d\varepsilon_x$ and $d\varepsilon_y^e = 0$, so $d\gamma^e = d\varepsilon_x/2$ on OA, BC and FO.

From Appendix J, for aluminum

$$-(\partial\mu/\partial T)_{V,\gamma^e} \simeq .14 \ kb/°C \ at \ 300°K.$$

From Eq. (75a), $V(\partial\mu/\partial T)$ is constant. For $V = V_0 = .37cc/g$, we have

$$-(16V/3)(\partial\mu/\partial T)_{\gamma^e, V} \simeq .27 \ kb \ cc/g°C.$$

For $\Gamma = 2$, $C_V = 8.4 \times 10^{-3}$ kb cc/g°C, we get

$$dS = (16.8 \times 10^{-3} \, d\varepsilon_x + .27 \, \gamma^e d\gamma^e) \ kb \ cc/g°C.$$

From 0 to A, $d\gamma^e = d\varepsilon_x/2$ and $\gamma^e = \varepsilon_x/2$, so at A

$$S(A) = 16.8 \times 10^{-3} \, \varepsilon_x(A) + .03375 \, \varepsilon_x^2(A) = \underline{-2.950 \times 10^{-4}}.$$

On AB, $d\gamma^e = 0$, so

$$S(B) = S(A) + 16.8 \times 10^{-3} \, [\varepsilon_x(B) - \varepsilon_x(A)] = -7.345 \times 10^{-4}$$

From B to C $d\gamma^e = d\varepsilon_x/2$ but $\gamma^e \neq \varepsilon_x/2$, so it is simpler to retain γ^e.

Thus

$$dS = 16.8 \times 10^{-3} \; d\epsilon_x + .135 \; d(\gamma^e)^2$$

$$S(C) = S(B) + 16.8 \times 10^{-3} \; [\epsilon_x(C) - \epsilon_x(B)]$$

$$+ .135[\gamma^{e^2}(C) - \gamma^{e^2}(B)] = \underline{-1.220 \times 10^{-4}}$$

From C to F $\quad d\gamma^e = 0$, so

$$S(F) = S(C) + 16.8 \times 10^{-3} \; [\epsilon_x(F) - \epsilon_x(C)] = \underline{3.175 \times 10^{-4}}$$

Finally at 0

$$S(0) = S(F) + 16.8 \times 10^{-3} \; [\epsilon_x(0) - \epsilon_x(F)]$$

$$+ .135 \; [(\gamma^e)^2 (0) - (\gamma^e)^2 (F)] = \underline{4.68 \times 10^{-9}} \simeq 0.0$$

Entropy of the Reservoir

$$S_R(A) = -S(A) = 2.950 \times 10^{-4}$$

$$S_R(B) = S_R(A) - (S_B - S_A) + W_p(B)/300 = \underline{9.429 \times 10^{-4}}$$

$$S_R(C) = S_R(B) - (S_C - S_B) = \underline{3.303 \times 10^{-4}}$$

$$S_R(F) = S_R(C) - [S_F - S_C] + [W_p(F) - W_p(C)]/2 = \underline{1.068 \times 10^{-4}}$$

$$S_R(0) = S_R^{\;F} - (S_0 - S_F) = \underline{4.243 \times 10^{-4}}$$

TABLE D-1. Values of mechanical and thermal variables at various points on the cycle of Fig. 4.   Calculated from Program "CYCLE." Columns A, B, etc. correspond to points A, B, etc. in Fig. 4.
First entry in each box: Isothermal cycle, T = 300°K
Third entry in each box: Isentropic cycle, S = 0
Fourth entry in each box: Adiabatic cycle

The entropy in parenthesis is obtained from exact integration as described in the first part of this Appendix.

	A	B	C	F	0
σ_x	-20.6206	-40.6000	.642331	20.6192	0.0
	(-20.6)	(-40.6)	(0.6)	(20.6)	(0.0)
	-20.5921	-40.6000	.461812	20.6454	0.0
	"	"	.380745	20.3785	"
σ_y	-10.6206	-30.6001	-9.35767	10.6193	-.000496
	(-10.6)	(-30.6)	(-9.4)	(10.6)	(0)
	-10.5921	-30.6001	-9.53819	10.6455	-.000089
	"	"	-9.61926	10.3786	-.096066
p	13.9540	33.9334	6.02433	-13.9526	.000330
	---	---	---	---	---
	13.9254	33.9334	6.20485	-13.9788	.000059
	"	"	6.28592	-13.7119	.064044
V	.363308	.353937	.367091	.376806	.369991
	---	---	---	---	---
	.363308	.353852	.367002	.376806	.369991
	"	.353837	.366987	.376736	.369987
ϵ_x	-.0182483	-.0443763	-.00787783	.0182470	-4×10^{-8}
	(-.01823)	(-.04439)	(-.00793)	(.01823)	(0.0)
	-.0182482	-.0446184	-.00812133	.0182478	1×10^{-8}
	"	-.0446597	-.00816250	.0180612	-1.1×10^{-5}
γ^e	-.00912418	-.00912418	.00912448	.00912448	$.9 \times 10^{-6}$
	(-.009113)	(-.009113)	(.009117)	(.009117)	(0.0)
	-.00912413	-.00912413	.00912403	.00912403	1×10^{-7}
	"	"	.00912406	.00912406	$.88 \times 10^{-4}$
ϵ_x^p	0.0	-.0174187	-.0174187	-2×10^{-6}	-2×10^{-6}
	(0.0)	(-.017428)	(-.017428)	(0.0)	(0.0)
	0.0	-.0175801	-.0175801	-6×10^{-7}	-6×10^{-7}
	"	-.0176076	-.0176076	-.000125060	-1.3×10^{-4}
ϵ_x^e	-.0182483	-.0269576	.00954075	.0182490	2×10^{-6}
	(-.01823)	(-.02694)	(.00952)	(.01823)	(0.0)
	-.0182482	-.0270382	.00945874	.0182484	6×10^{-7}
	"	-.0270520	.00944509	.0181863	1.1×10^{-4}

73

	A	B	C	F	0
ϵ_y^e	0.0	-.00870935	-.00870935	-1×10^{-6}	-1×10^{-6}
	(0.0)	(-.008714)	(-.008714)	(0.0)	(0.0)
	0.0	-.00879006	-.00879006	-3×10^{-7}	-3×10^{-7}
	"	-.00880379	-.00880379	-6.25×10^{-4}	-6.2×10^{-5}
S	-.000306570	-.000745514	-.000132340	.000306558	7×10^{-9}
	(-.000295)	(-.000735)	(-.000122)	(.000317)	(5×10^{-9})
	0.0	0.0	0.0	0.0	0.0
	"	+.000195374	.000195374	.000406573	.000406573
W_p	0.0	.0624640	.0624640	.127240	.127240
	(0.0)	(.0625)	(.0625)	(.1273)	(.1273)
	0.0	.0630353	.0630353	.128409	.128409
	"	.0631326	.0631326	.128138	.128138
T	300	300	300	300	300
	(300)	(300)	(300)	(300)	(300)
	311.145	327.989	304.891	289.223	299.966
	"	335.734	312.090	303.679	314.849

Program CYCLE

In the general case where λ, μ, Y may be functions of density and temperature, numerical integration of the preceding equations may be required. This can be accomplished with the FORTRAN program described here. With it one can obtain values of the variables listed above for isothermal, isentropic or adiabatic cycles. The last of these is a cycle only in the sense that stress returns to its original value. The other variables do not because plastic work is retained in the sample.

Results of integrations for conditions described in the first part of this Appendix are listed in Table D-1 for the vertices ABCFO. For isothermal and isentropic cases the cyclic variables do not quite return to zero. These integrations are accomplished by a direct, approximately second-order integration scheme. The listed values were obtained with $d\sigma_x = -.406$ kb; reducing it to a half or a third of this value did not significantly improve the closing values. For greater accuracy a better integration scheme would be required.

The following assumptions are made about the equation of state:
bulk modulus:

$$K(T,V) = K_0 + bp + \delta(T-T_0) + gp(T-T_0)$$

where K_0 = isothermal bulk modulus at $T = T_0$, $p = 0$

b = polytropic coefficient, i.e. $p(T_0,V) = (K_0/b)[(V_0/V)^b -1]$

δ = $\beta - \alpha K_0 b$

α = thermal expansion coefficient, $(1/V)(\partial V/\partial T)_p$

β = $C_V(\Gamma_0 + \Gamma_1)/V_0$

C_V = specific heat at constant volume

V_0 = specific volume at $T = T_0$, $p = 0$

Γ = Gruneisen parameter = $\Gamma_0 + \Gamma_1(1- V/V_0)$

g = $(\beta/3K_0) (4-1/b) - \alpha b^2$

75

$$\mu = \mu_0 + b_1 (T/V - T_0/V_0)$$

T_0 = const. is a reference temperature

For aluminum these constants are approximately

$$\Gamma_0 = 2.14, \quad \Gamma_1 = -3.2, \quad C_V = 8.4 \times 10^{-3} \text{ kb cc/g}^\circ K, \quad V_0 = .37 \text{ cc/g}$$

$$\alpha = .75 \times 10^{-4}/^\circ K, \quad K_0 = \lambda + 2\mu/3 = 764.67 \text{ kb}$$

$$\lambda = 582 \text{ kb}, \quad \mu_0 = 274 \text{ kb}, \quad T_0 = 300^\circ K$$

$$\beta = -.024065 \text{ kb}/^\circ K, \quad \delta = -.26895 \text{ kb}/^\circ K$$

$$g = -1.40697 \times 10^{-3}/^\circ K, \quad b = 4.27$$

Provision is also made for introducing work-hardening in an elementary way. The yield strength can vary with plastic work according to the relation

$$Y = Y_0 + A[1 - \exp(W_p/W_0)]$$

where Y_0, A and W_0 are constants.

Comments on the Program Listing

IC   is the integer index which determines whether the computation will be made under isothermal, isentropic or adiabatic conditions.

N   The stress increment used in integration is σ_x^B/N where $-\sigma_x^B$ is the maximum compressive stress to be reached.

MM   This is the number of cycles to be calculated. For this run two cycles are calculated. IC = 2 for the first and IC = 3 for the second. IC is incremented by card number 0184 (left hand column).

VØØ   = V_0

TØ   = T_0

KØ   = K_0

B   = b

GAMO   = Γ_0

GAM1   = Γ_1

CV   = C_V

MUØ   = μ_0

BETA   = β

DELTA   = δ

SGAM   = g

AMU   = b_1

YØ   = initial value of the yield stress

A   = A

WØ   = W_0

SIGXB = σ_x^B = -(compressive stress at which stress cycle is reversed)

MCOUNT = integer index which counts the number of cycles. It is incremented by 1 at the end of each cycle.

V(J)   = specific volume for J^{th} increment in stress

EPX $= \varepsilon_x$

EPY $= \varepsilon_y$

EXE $= \varepsilon_x^e$

EYE $= \varepsilon_y^e$

EPXP $= \varepsilon_x^p$

EPYP $= \varepsilon_y^p$

GAMME $= \gamma^e$

S(J) $=$ entropy

K $=$ isothermal bulk modulus $= K$

MU $= \mu$

LAM $= \lambda$

YB $= \overline{Y}$

WPB $= \overline{W}_p$

TB $= \overline{T}$ These quantities indicated by B or $^-$ are averages of values at the last J and the next J. Their use makes the integration scheme approximately second order.

DY $= dY$

ISTOP, IFLAG   are control indices used for stopping the calculation and calculating the end of the cycle.

IY   is an index used to guide the computation through the elastic branch (IY=1) or elastic-plastic branch (IY=2).

JA = 1 indicates region FOA of Fig. 4.

   = 2 indicates region AB   of Fig. 4.

   = 3 indicates region BC   of Fig. 4.

   = 4 indicates region CF   of Fig. 4.

JM   is the index which directs the computation to the next branch of the cycle.

IS   is an index used to calculate the end points (A, B, C, F, O) of each branch.

GAM $= \Gamma$

The program contains a subroutine for use with a CALCOMP plotter. It is by-passed in this listing, but it can be engaged by removing card number 0233, "G∅ T∅ 99."

Following the listing is sample output for the two cases shown in Table D-1 for isentropic and adiabatic compression.

PROGRAM "CYCLE"

```
0001            DIMENSION SIGX(1500),SIGY(1500),EPX(1500),EPY(1500),EXE(1500)       R001
0002            DIMENSION EYE(1500),EPXP(1500),EPYP(1500),WP(1500),Y(1500)          R002
0003            DIMENSION P(1500),V(1500),J1(1500)                                  R003
0004            DIMENSION T(1000)
0005            DIMENSION S(1000), GAMME(1000)
0006            REAL K, KO, MU, LAM, MUO, MU1, MU2
0007            REAL J1,LENGX,LENGY                                                 R005
0008            INTEGER PENP                                                        R005AAA
0009            DATA XEPX/' EPX'/,XEXE/' EXE'/,XEYE/' EYE'/                         R005AA
0010            DATA XJ1/' J  '/,XSIGX/'SIGX'/,XSIGY/'SIGY'/,XEPXP/'EPXP'/          R005A
      C
      C KO,K,LAM,MU,MUO        ARE IN KILOBARS
      C B IS DIMENSIONLESS
      C BETA IS IN KILOBARS/DEGREE C, BEAT=CV*(GAMO+GAM1)/VOO
      C CV IS IN KB-CC/GRAM-DEGREE C,
      C V,VOO ARE IN CC/G
      C GAM,GAMO,GAM1 ARE DIMENSIONLESS
      C DELTA IS IN KILOBARS/DEGREE C
      C DELTA=-(ALPHA*KO*B-BETA)
      C ALPHA= VOLUME EXPANSION COEFFICIENT,(DEG C)**-1,=.75E-4 FOR AL
      C K=ISOTHERMAL BULK MODULUS
      C LAM AND MU ARE LAME' CONSTANTS
      C SGAM IS IN (DEG C)**-1, SGAM=(4*BETA/3KO-BETA/(3*KO*B)-ALPHA*(BETA)**2
      C IY=1 IF COMPRESSION (OR EXPANSION) IS ELASTIC
      C IY=2 IF COMPRESSION IS ELASTIC-PLASTIC
      C IC=1 FOR ISOTHERMAL PROCESS
      C IC=2 FOR ISENTROPIC PROCESS
      C IC=3 FOR ADIABATIC PROCESS
      C     CONSTANTS                                                               R006
      C     ALUMINUM                                                                R007
      C
0011            IC=1
0012            N=100
0013            MM=2
```

```
0014          VOO=.370                                              R010
0015          TO=300.
0016          KO=764.67
0017          B=0
0018          GAMO=2.0
0019          GAM1=0.0
0020          CV=8.4E-3
0021          MUO=274.
0022          BETA=-.024065
0023          DELTA=-.26895
0024          SGAM=-1.407E-3
0025          AMU=0.0
0026          YO=10.
0027          A=0.0                                                 R015
0028          WO=.34                                                R016
0029          SIGXB=-40.6                                           R017
0030          Q=FLOAT(N)
0031          DSIGX=SIGXB/Q
0032          MCOUNT=1                                              R019
       C
       C
       C      INITIAL CONDITIONS                                    R020
       C
0033          V(1)=VOO                                              R021
0034          P(1)=0.0                                              R022
0035          SIGX(1)=0.0                                           R023
0036          SIGY(1)=0.0                                           R024
0037          EPX(1)=0.0                                            R025
0038          EPY(1)=0.0                                            R026
0039          EXE(1)=0.0                                            R027
0040          EYE(1)=0.0                                            R028
0041          EPXP(1)=0.0                                           R029
0042          EPYP(1)=0.0                                           R030
0043          GAMME(1)=0.0
0044          WP(1)=0.0                                             R031
0045          T(1)=TO
0046          S(1)=0
0047          Y(1)=YO                                               R035
```

```
0048            K=KO                                                        R032
0049            MU=MUO
C050            LAM=K-(2./3.)*MU
0051            YB=YO
0052            WPB=WP(1)
0053            TB=T(1)
0054            DY=0.
0055            VB=VOO
0056            ISTOP=1
0057            IFLAG=1
0058        400 CONTINUE
          C
          C     ALONG OA                                                    R036
0059            WRITE(6,101)
0060        101 FORMAT(4X,'J',8X,'SIGX',11X,'SIGY',13X,'P',14X,'Y',14X,'V',11X,
               +'EPX',11X,'GAMME')
          C
0061            WRITE(6,102)
0062            J=1
0063        301 CONTINUE                                                    R037
0064            IY=1
0065            JA=1
0066            JM=2
0067            IS=1
0068            GO TO 201
0069        302 CONTINUE
0070            IY=2
0071            JA=2
0072            JM=3
0073            IS=1
0074            GO TO 201
0075        303 CONTINUE
0076            IY=1
0077            JA=3
0078            JM=4
```

85

```
0079          DSIGX=-DSIGX
0080          IS=1
0081          GO TO 201
0082      304 CONTINUE
0083          IY=2
0084          JA=4
0085          JM=1
0086          IS=1
0087          DSIGX=-DSIGX
0088          SIGXT=0.0
0089      201 CONTINUE
0090        1 SIGX(J+1)=SIGX(J)+DSIGX                                    R039
      C
0091          IF(IFLAG.EQ.2.AND.SIGX(J+1).LT.0.0) GO TO 2
0092          GO TO 3
0093        2 CONTINUE
0094          DSIGX=-SIGX(J)
0095          IFLAG=1
0096          ISTOP=2
0097          WRITE(6,110)
0098      110 FORMAT(2X,'NEXT VALUES COMPLETE THE CYCLE')
0099          GO TO 1
0100        3 CONTINUE
      C   COMPUTE SIGY(J+1)
      C
0101          DO 8 M=1,3                                                 R041
0102          GO TO (41,42),IY
0103       41 DSIGY=LAM*DSIGX/(LAM+2.*MU)
0104          GO TO 43
0105       42 DSIGY=DSIGX+DY
0106          IF(JA.EQ.4) DSIGY=DSIGX-DY
0107       43 CONTINUE
0108          SIGY(J+1) = SIGY(J)+DSIGY
0109          P(J+1)=-(SIGX(J+1)+2.*SIGY(J+1))/3.
0110          PB=(P(J+1)+P(J))/2.
0111          DP=P(J+1)-P(J)                                             R046
0112          K=K0+B*PB+(DELTA+SGAM*PB)*(TB-TO)
```

96

```
0113          DV=-VB*DP/K                                              R048
0114          DEPX=-DP/K                                               R047
0115          V(J+1)=V(J)+DV                                           R049
0116          VB=V(J)+DV/2.
0117          MU=MU0+AMU*(TB/VB-TO/VOO)
0118          LAM=K-(2./3.)*MU
       C
       C   COMPUTE PLASTIC WORK AND Y
       C
0119          GO TO (51,52),IY
0120       51 DWP=0.
0121          DY=0.
0122          GO TO 53
0123       52 CONTINUE
0124          IF(JA.EQ.4)YB=-YB
0125          IF(JA.EQ.4)DY=-DY
0126          DWP=-(2.*VB*YB/3.)*(DEPX+DY/(2.*MU))
0127          DY=(A/WO)*DWP*EXP(-WPB/WO)
0128       53 CONTINUE
0129          WP(J+1)=WP(J)+DWP                                        R086
0130          WPB=(WP(J+1)+WP(J))/2.                                   R087
0131          Y(J+1)=Y(J)+DY                                           R085
0132          YB=(Y(J+1)+Y(J))/2.                                      R140
       C
       C   COMPUTE GAMMA-ELASTIC
       C
0133          DGAMME=(DSIGX-DSIGY)/(4.*MU)
0134          GAMME(J+1)=GAMME(J)+DGAMME
0135          GAMMEB=GAMME(J)+DGAMME/2.
       C
       C COMPUTES S AND T
```

```
        C
0136            GAM=GAMO+GAM1*(1.-VB/VOO)
0137            GO TO (31,32,33),IC
0138         31 DT=0
0139            DS=GAM*CV*DV/VB-16.*GAMMEB*AMU*DGAMME/3.
0140            GO TO 34
0141         32 DS=0
0142            DT=-GAM*TB*DV/VB+16.*GAMMEB*TB*AMU*DGAMME/(3.*CV)
0143            GO TO 34
0144         33 DS=DWP/TB
0145            DT=TB*DS/CV-GAM*TB*DV/VB+16.*GAMMEB*TB*AMU*DGAMME/(3.*CV)
0146         34 T(J+1)=T(J)+DT
0147            S(J+1)=S(J)+DS
0148            TB=T(J)+DT/2.
0149        102 FORMAT(15X,'EPXP',12X,'EPXE',10X,'EPYE',13X,'S',15X,'WP',10X,'T')
0150          8 CONTINUE
        C
        C
0151            GO TO (61,62),IY
        C
        C COMPRESSION IS ELASTIC
        C
0152         61 DEPXP=0.
0153            DEPXE=DEPX
0154            DEPYP=0.
0155            DEPYE=0.
0156            GO TO 63
        C
        C COMPRESSION IS ELASTIC-PLASTIC
        C
0157         62 CONTINUE
0158            DEPXP=(2./3.)*(DEPX+DY/(2.*MU))
0159            DEPXE=DEPX-DEPXP
0160            DEPYP=-DEPXP/2.
0161            DEPYE=-DEPYP
0162         63 CONTINUE
```

R093

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0163          EPX(J+1)=EPX(J)+DEPX                                                R091
0164          EPXP(J+1)=EPXP(J)+DEPXP                                             R090
0165          EXE(J+1)=EXE(J)+DEPXE                                               R094
0166          EPYP(J+1)=EPYP(J)+DEPYP
0167          EYE(J+1)=EYE(J)+DEPYE                                               R096
0168          J=J+1                                                              R097
0169          WRITE(6,104)J,SIGX(J),SIGY(J),P(J),Y(J),V(J),EPX(J),GAMME(J)
0170      104 FORMAT(2X,I4,5F15.8,2E15.6)
0171          WRITE(6,105)EPXP(J),EXE(J),EYE(J),S(J),WP(J),T(J)
0172      105 FORMAT(10X,5E15.6,F15.8)
0173          IF(J.GT.999)GO TO 21
0174          IF(IS.EQ.2)DSIGX=SIGXB/Q
0175          IF(IS.EQ.2) GO TO 76
0176          GO TO (71,72,73,74),JA
0177       71 CONTINUE
0178          IF(ISTOP.EQ.1)GO TO 5
0179          IF (MCOUNT.GT.MM)GO TO 21
0180        4 CONTINUE
0181          J=1
0182          ISTOP=1
0183          IFLAG=1
0184          IC=IC+1
0185          IF(IC.GT.3)GO TO 21
0186          DSIGX=SIGXB/Q
0187          GO TO 400
0188        5 CONTINUE
0189          IF ((SIGX(J)-SIGY(J)).GT.-Y(J)) GO TO 201
0190          ZN=SIGX(J)-SIGY(J)
0191          ZO=SIGX(J-1)-SIGY(J-1)
0192          DZ=ZN-ZO
0193          DSIGX=-(DSIGX/DZ)*(Y(J-1)+ZO)/(1.-DY/DZ)
0194          J=J-1
0195          IS=IS+1
0196          WRITE(6,106)
0197      106 FORMAT(10X,'NEXT VALUES ARE FOR POINT A')
0198          GO TO 201
0199       72 IF(SIGX(J).GT.SIGXB) GO TO 201
0200          DSIGX=SIGXB-SIGX(J-1)
```

68

```
0201            J=J-1
0202            IS=IS+1
0203            WRITE(6,107)
0204        107 FORMAT(10X,'NEXT VALUES FOR POINT B')
0205            GO TO 201
0206         73 IF((SIGX(J)-SIGY(J)).LT.Y(J)) GO TO 201
0207            ZN=SIGX(J)-SIGY(J)
0208            ZO=SIGX(J-1)-SIGY(J-1)
0209            DZ=ZN-ZO
0210            DSIGX=(DSIGX/DZ)*(Y(J-1)-ZO)/(1.-DY/DZ)
0211            J=J-1
0212            IS=IS+1
0213            WRITE(6,108)
0214        108 FORMAT(10X,'NEXT VALUESFOR POINT C')
0215            GO TO 201
0216         74 IF(EPX(J).LT.0.0) GO TO 201
0217            IF(EPX(J-1).LT.0.0) DEPX=EPX(J)
0218            DSIGXT=(LAM+2.*MU)*DEPX
0219            SIGXT=SIGXT+DSIGXT
0220            IF(SIGXT.LT.SIGX(J)) GO TO 201
0221            DSIGX=(DSIGX/DSIGXT)*(SIGX(J-1)-SIGXT+DSIGXT)/(1.-DSIGX/DSIGXT)
0222            J=J-1
0223            IS=IS+1
0224            WRITE(6,109)
0225        109 FORMAT(10X,'NEXT VALUES FOR POINT F')
0226            IFLAG=2
0227            MCOUNT=MCOUNT+1                                                      R160
0228            GO TO 201
0229         76 CONTINUE
0230            GO TO (301,302,303,304),JM
0231         21 CONTINUE                                                            R162
0232         55 CONTINUE
0233            GO TO 99
0234         56 CONTINUE
0235            DO 20 K2=1,J                                                        R163
```

```
0236        20 J1(K2)=K2                                                         R164
0237           ZERO=1.0                                                          R165
0238           LENGX=4.5                                                         R166
0239           LENGY=LENGX                                                       R167
0240           NCHARX=4                                                          R168
0241           NCHARY=NCHARX                                                     R169
0242           PLEND=7.0                                                         R170
0243           PENP=-3                                                           R171
0244           ORIGIN=0.0                                                        R172
0245           FACT=1.0                                                          R173
0246           CALL PLOTIT(J1,SIGX,LENGX,LENGY,NCHARX,NCHARY,XJ1,XSIGX,PLEND,ZEROR174
               C,J,PENP,ORIGIN,FACT)                                             R175
0247           CALL PLOTIT(J1,SIGY,LENGX,LENGY,NCHARX,NCHARY,XJ1,XSIGY,PLEND,ZEROR176
               C,J,PENP,ORIGIN,FACT)                                             R177
0248           CALL PLOTIT(J1,EPXP,LENGX,LENGY,NCHARX,NCHARY,XJ1,XEPXP,PLEND,ZEROR178
               C,J,PENP,ORIGIN,FACT)                                             R179
0249           CALL PLOTIT(J1,EPX,LENGX,LENGY,NCHARX,NCHARY,XJ1,XEPX,PLEND,ZERO,JR180
               C,PENP,ORIGIN,FACT)                                               R181
0250           CALL PLOTIT(J1,EXE,LENGX,LENGY,NCHARX,NCHARY,XJ1,XEXE,PLEND,ZERO,JR182
               C,PENP,ORIGIN,FACT)                                               R183
0251           PENP=999                                                          R184
0252           PLEND=10.0                                                        R185
0253           CALL PLOTIT(J1,EYE,LENGX,LENGY,NCHARX,NCHARY,XJ1,XEYE,PLEND,ZERO,JR186
               C,PENP,ORIGIN,FACT)                                               R187
0254        99 STOP                                                              R188
0255           END                                                              R189
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```
0001          SUBROUTINE PLOTIT(X,Y,LENGX,LENGY,NCHARX,NCHARY,TITLEX,TITLEY,PLENS100
              CD,ZERO,NPTS,PENP,ORIGN,FCTR)                                       S101
0002          DIMENSION IBUF(1500),X(1500),Y(1500)                               S102
0003          REAL*8 TITLEX,TITLEY                                               S103
0004          REAL LENGX,LENGY                                                   S104
0005          INTEGER PENP                                                       S105
0006          DATA ENTER/1./                                                     S106
      C     INSTRUCTIONS FOR USING THE PLOT SUBROUTINE:
      C
      C
      C           THE TITLES FOR THE PLOTS MUST BE LITERAL CONSTANTS NO
      C     LONGER THAN EIGHT CHARACTERS IN LENGTH AND MUST BE DECLARED
      C     REAL*8.
      C           FOR THE CASE OF TOTALLY POSITIVE DATA THE SCALE ROUTINE
      C     WILL RETURN A NON-ZERO ORIGIN FOR ONE OR BOTH AXES.  IF A
      C     ZERO ORIGIN IS DESIRED FOR BOTH AXES THE VARIABLE ORIGN SHOULD
      C     BE SET TO ZERO AND FOR NON-ZERO ORIGINS ORIGN SHOULD BE SET TO
      C     ONE.
0007          CONTINUE
      C           IN CALLING THE SUBROUITNE THE VALUE ASSIGNED TO THE VAR-
      C     IABLE ZERO SHOULD BE NO SMALLER THAN .5 SINCE ANYTHING SMALLER
      C     WOULD NOT ALLOW ROOM TO PRINT THE TITLE FOR THE X-AXIS.
      C           CARE MUST BE TAKEN IN CHOOSING THE VALUES FOR THE VAR-
      C     IABLE LENGY.  THE SUM OF LENGY AND ZERO SHOULD NOT EXCEED 10.
      C     THE WIDTH OF THE PLOTTING PAPER IS ONLY 11 INCHES AND THE
      C     VALUE FOR ZERO TELLS THE PLOTTER HOW FAR ABOVE THE BOTTOM OF
      C     THE PAPER TO BEGIN DRAWING THE Y-AXIS, THEREFORE THE AXIS MUST
      C     BE SHORT ENOUGH TO FIT ON THE PAPER WITHOUT HITTING THE LIMIT
      C     SWITCH ON THE TOP.
      C           THE VARIABLE FCTR IS USED TO CHANGE THE SIZE OF THE PLOTS
      C     WITHOUT CHANGING THE LENGTH OF THE AXES.  IT PRODUCES A PLOT
      C     WHICH IS A SCALED REPRESENTATION OF THE NORMAL PLOT.  IF FCTR
      C     IS NOT EQUAL TO ONE CARE MUST BE TAKEN TO INSURE THE NEW PLOT
      C     WILL NOT BE TOO LARGE TO FIT ON THE PAGE OR TOO CLOSE TO THE
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92

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C         BOTTOM LIMIT SWITCH.  A PLOT MADE VERY SMALL BY THE USE OF THE
C         FCTR PARAMETER MAY NOT BE READABLE, THEREFORE THE USE OF FCTR
C         SHOULD BE LIMITED.
C              THE VARIABLE PLEND IS THE NUMBER OF INCHES FROM THE
C         ORIGIN THE PEN WILL MOVE IN THE X DIRECTION AT THE END OF THE
C         PLOT.  THEREFORE THE VALUE ASSIGNED TO PLEND SHOULD BE THE
C         LENGTH OF THE X-AXIS PLUS THE NUMBER OF INCHES PAST IT THE
C         NEW PLOT IS TO BEGIN.  THE VALUE ASSIGNED TO PLEND IN THE
C         FINAL CALL SHOULD BE AT LEAST THE VARIABLE LENGX + 5 TO ALLOW
C         THE PAPER TO BE TORN OFF THE PLOTTER WITHOUT AFFECTING THE
C         PLOT.
0008      CONTINUE
C              THE VARIABLE PENP SHOULD BE ASSIGNED THE VALUE -3 FOR
C         ALL CALLS TO THE SUBROUTINE EXCEPT THE FINAL CALL.  FOR THE
C         FINAL CALL THE VALUE 999 SHOULD BE GIVEN TO PENP.  THESE
C         VALUES MERELY RAISE THE PEN WHILE MOVING TO THE NEW ORIGIN,
C         OR FOR THE CASE WHEN PENP=999, WHILE MOVING TO THE END OF THE
C         PAPER.
C
0009      CONTINUE
C
C         X= ARRAY CONTAINING X COORDINATES
C         Y= ARRAY CONTAINING Y COORDINATES
C         LENGX= LENGTH OF X-AXIS
C         LENGY= LENGTH OF Y-AXIS

C         NCHARX= NUMBER OF CHARACTERS IN TITLE OF X-AXIS
C         NCHARY= NUMBER OF CHARACTERS IN TITLE OF Y-AXIS
C         TITLEX= TITLE TO BE GIVEN X-AXIS
C         TITLEY= TITLE TO BE GIVEN Y-AXIS
C         PLEND= DISTANCE PEN WILL MOVE AT END OF PLOT
C         ZERO= DISTANCE ABOVE BOTTOM OF PAGE PEN SHOULD BE MOVED TO
C              BEGIN Y-AXIS
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93

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C        NPTS= NUMBER OF POINTS + 2 IN ARRAYS TO BE PLOTTED ( NUMBER
C               OF X POINTS SHOULD EQUAL NUMBER OF Y POINTS.   THE TWO
C               EXTRA PLACES IN THE ARRAYS ARE USED IN THE PLOTTING
C               ROUTINE FOR SCALING)
C        PENP= POSITION OF PEN AT END OF PLOT (NORMALLY EITHER -3 OR
C               999)
C        ORIGN= 0.0 FOR ZERO ORIGINS AND 1.0 FOR NON-ZERO ORIGINS
C        FCTR= RATIO OF NEW PLOT SIZE TO THE NORMAL PLOT SIZE.   A VALUE
C               OF 1.0 GIVES PLOTS OF NORMAL SIZE.   ANYTHING GREATER
C               THAN ONE INCREASES THE SIZE OF THE PLOT WHILE ANYTHING
C               LESS THEN DECREASES THE PLOT SIZE
C
      IF(ENTER.NE.1) GO TO 1                                            S107
      CALL PLOTS(IBUF,1000)                                            S108
    1 ENTER=2.0                                                        S109
      CALL PLOT(0.0,-11.0,-3)                                          S110
      CALL FACTOR(FCTR)                                                S110A
      CALL PLOT(0.0,ZERO,-3)                                           S111
      CALL SCALE(X,LENGX,NPTS,1)                                       S112
      CALL SCALE(Y,LENGY,NPTS,1)                                       S113
      XPAGE=0.0                                                        S113A
      YPAGE=0.0                                                        S113B
      IF(Y(NPTS+1)) 10,20,30                                           S114
   20 IF(X(NPTS+1)) 21,22,23                                           S115
   22 CALL AXIS(0.0,YPAGE,TITLEX,-NCHARX,LENGX,0.0,X(NPTS+1),X(NPTS+2)) S116
      CALL AXIS(XPAGE,0.0,TITLEY,NCHARY,LENGY,90.0,Y(NPTS+1),Y(NPTS+2)) S117
      CALL LINE( X,Y,NPTS,1,0,0)                                       S118
      CALL PLOT(PLEND,-11.0,PENP)                                      S119
      GO TO 99                                                         S120
   21 XPAGE=-X(NPTS+1)/X(NPTS+2)                                       S121
      GO TO 22                                                         S123
   23 IF(ORIGN.NE.0.0) GO TO 22                                        S127
      XMAX=X(NPTS+1)+LENGX*X(NPTS+2)                                   S128
      X(NPTS+1)=0.0                                                    S129
      X(NPTS+2)=XMAX/LENGX                                             S130
      GO TO 22                                                         S131
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94

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0034        10 YPAGE=-Y(NPTS+1)/Y(NPTS+2)                        S132
0035           IF(X(NPTS+1)) 11,12,13 ·                          S133
0036        12 GO TO 22                                          S135
C037        11 XPAGE=-X(NPTS+1)/X(NPTS+2)                        S136
0038           GO TO 22                                          S137
0039        13 IF(ORIGN.NE.0.0) GO TO 12                        S138
0C40           XMAX=X(NPTS+1)+LENGX*X(NPTS+2)                    S139
0041           X(NPTS+1)=0.0                                     S140
0042           X(NPTS+2)=XMAX/LENGX                             S141
0C43           GO TO 22                                          S142
0044        30 IF(X(NPTS+1)) 31,32,33                           S158
0045        32 IF(ORIGN.NE.0) GO TO 22                          S159
0046        34 YMAX=Y(NPTS+1)+LENGY*Y(NPTS+2)                    S160
0C47           Y(NPTS+1)=0.0                                     S161
0C48           Y(NPTS+2)=YMAX/LENGY                             S162


0049           GO TO 22                                          S163
0050        31 XPAGE=-X(NPTS+1)/X(NPTS+2)                       S164
0051           GO TO 32                                          S166
0052        33 IF(ORIGN.NE.0.0) GO TO 22                        S170
0053           XMAX=X(NPTS+1)+LENGX*X(NPTS+2)                   S171
0054           X(NPTS+1)=0.0                                     S172
0C55           X(NPTS+2)=XMAX/LENGX                             S173
C056           GO TO 34                                          S174
0057        99 RETURN                                            S175
CC58           END                                               S176
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95

J	SIGX	SIGY	P	Y	V	EPX	GAMME
	EPXP	EPXE	EPYE	S	WP	T	
2	-0.40599990	-0.20910841	0.27473891	10.00000000	0.36986703	-0.359291E-03	-0.179646E-03
	0.0	-0.359291E-03	0.0	-0.603608E-05	0.0	300.00000000	
3	-0.81199980	-0.41821682	0.54947758	10.00000000	0.36973411	-0.718581E-03	-0.359291E-03
	0.0	-0.718581E-03	0.0	-0.120722E-04	0.0	300.00000000	

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J	SIGX	SIGY	P	Y	V	EPX	GAMME
51	-20.29986572	-10.45541763	13.73689747	10.00000000	0.36341119	-0.179644E-01	-0.898225E-02
	0.0	-0.179644E-01	0.0	-0.301802E-03	0.0	300.00000000	
52	-20.70585632	-10.66452599	14.01163197	10.00000000	0.36328059	-0.183237E-01	-0.916189E-02
	0.0	-0.183237E-01	0.0	-0.307838E-03	0.0	300.00000000	

NEXT VALUES ARE FOR POINT A

J	SIGX	SIGY	P	Y	V	EPX	GAMME
52	-20.62063599	-10.62062836	13.95396423	10.00000000	0.36330801	-0.182483E-01	-0.912418E-02
	0.0	-0.182483E-01	0.0	-0.306570E-03	0.0	300.00000000	
53	-21.02662659	-11.02662754	14.35995960	10.00000000	0.36311513	-0.187792E-01	-0.912418E-02
	-0.353961E-03	-0.184252E-01	-0.176981E-03	-0.315490E-03	0.128563E-02	300.00000000	

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J	SIGX	SIGY	P	Y	V	EPX	GAMME
101	-40.51417542	-30.51428223	33.84757996	10.00000000	0.35397691	-0.442640E-01	-0.912418E-02
	-0.173439E-01	-0.269202E-01	-0.867194E-02	-0.743629E-03	0.621992E-01	300.00000000	
102	-40.92016602	-30.92027283	34.25357056	10.00000000	0.35378897	-0.447950E-01	-0.912418E-02
	-0.176978E-01	-0.270972E-01	-0.884892E-02	-0.752549E-03	0.634518E-01	300.00000000	

NEXT VALUES FOR POINT B

J	SIGX	SIGY	P	Y	V	EPX	GAMME
102	-40.59999084	-30.60009766	33.93339539	10.00000000	0.35393715	-0.443763E-01	-0.912418E-02
	-0.174187E-01	-0.269576E-01	-0.870935E-02	-0.745514E-03	0.624640E-01	300.00000000	
103	-40.19398499	-30.39097595	33.65864563	10.00000000	0.35406429	-0.440170E-01	-0.894453E-02
	-0.174187E-01	-0.265983E-01	-0.870935E-02	-0.739478E-03	0.624640E-01	300.00000000	

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96

J	SIGX	SIGY	P	Y	V	EPX	GAMME
	EPXP	EPXE	EPYE	S	WP	T	
203	0.40636444	-9.47920322	6.18401051	10.00000000	0.36701411	-0.808666E-02	0.902007E-02
	-0.174187E-01	0.933193E-02	-0.870935E-02	-0.135848E-03	0.624640E-01	300.00000000	
204	0.81236434	-9.27009392	5.90926552	10.00000000	0.36714596	-0.772735E-02	0.919972E-02
	-0.174187E-01	0.969123E-02	-0.870935E-02	-0.129812E-03	0.624640E-01	300.00000000	

NEXT VALUES FOR POINT C

J	SIGX	SIGY	P	Y	V	EPX	GAMME
204	0.64233136	-9.35766888	6.02433205	10.00000000	0.36709070	-0.787783E-02	0.912448E-02
	-0.174187E-01	0.954075E-02	-0.870935E-02	-0.132340E-03	0.624640E-01	300.00000000	
205	1.04833126	-8.95166874	5.61833191	10.00000000	0.36728561	-0.734688E-02	0.912448E-02
	-0.170647E-01	0.971773E-02	-0.853236E-02	-0.123420E-03	0.637637E-01	300.00000000	

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J	SIGX	SIGY	P	Y	V	EPX	GAMME
253	20.53619385	10.53631401	-13.86960316	10.00000000	0.37676501	0.181385E-01	0.912448E-02
	-0.743989E-04	0.182128E-01	-0.371640E-04	0.304734E-03	0.126967E 00	300.00000000	
254	20.94218445	10.94231319	-14.27559853	10.00000000	0.37696511	0.186694E-01	0.912448E-02
	0.279562E-03	0.183898E-01	0.139817E-03	0.313654E-03	0.128301E 00	300.00000000	

NEXT VALUES FOR POINT F

J	SIGX	SIGY	P	Y	V	EPX	GAMME
254	20.61920166	10.61933422	-13.95262146	10.00000000	0.37680590	0.182470E-01	0.912448E-02
	-0.202062E-05	0.182490E-01	-0.974804E-06	0.306558E-03	0.127240E 00	300.00000000	
255	20.21319580	10.41022491	-13.67788124	10.00000000	0.37667054	0.178877E-01	0.894483E-02
	-0.202062E-05	0.178897E-01	-0.974804E-06	0.300521E-03	0.127240E 00	300.00000000	

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J	SIGX	SIGY	P	Y	V	EPX	GAMME
303	0.72513318	0.37298089	-0.49036467	10.00000000	0.37022889	0.641669E-03	0.321765E-03
	-0.202062E-05	0.643598E-03	-0.974804E-06	0.107875E-04	0.127240E 00	300.00000000	
304	0.31913328	0.16387248	-0.21562606	10.00000000	0.37009585	0.282378E-03	0.142119E-03
	-0.202062E-05	0.284308E-03	-0.974804E-06	0.475146E-05	0.127240E 00	300.00000000	

NEXT VALUES COMPLETE THE CYCLE

J	SIGX	SIGY	P	Y	V	EPX	GAMME
305	0.0	-0.00049567	0.00033045	10.00000000	0.36999130	-0.393484E-07	0.910179E-06
	-0.202062E-05	0.189035E-05	-0.974804E-06	0.684031E-08	0.127240E 00	300.00000000	

APPENDIX E

Gibbs Relation

Consider an element of mass within which may operate various irreversible processes. The element is supposed to be in a stable or equilibrium state; if it is left undisturbed it remains unchanged. We also suppose that temperature T, specific entropy S, and specific internal energy E can be assigned to it. External forces are applied at a particular instant and work is done on the element in the amount dW. Heat of amount dQ is also added. These external influences set in motion irreversible processes, some of which operate immediately and some of which may operate over an extended time interval. After a long time, during which the element is insulated from further heat exchange, the element is assumed to come to a new equilibrium state which differs from the first by internal energy dE, temperature dT and entropy dS. We suppose that the only net effect of adding dQ and dW is to increase the internal energy:

$$dE = dQ + dW \qquad (E1)$$

We further suppose that some method exists for assessing the internal changes in the sample, i.e. for partitioning the change dE into changes in thermal energy, TdS, and coherent energy, say dW_e. For the purposes of Section IV this is accomplished by writing the total work increment as the sum of an elastic and plastic term and assuming that the elastic work is reversible:

$$dW = dW_e + dW_p \qquad (E2)$$

$$dE = TdS + dW_e \qquad (E3)$$

Combining Eqs. (E1) - (E3) yields the Gibbs relation

$$TdS = dW_p + dQ \qquad (E4)$$

Properties of the material and properties postulated to this point imply that there also exist other thermo-dynamic functions, e.g. the Helmholtz free energy,

99

$$A = E - TS$$

$$dA = -SdT + dW_e$$

$$= -SdT + dW - dW_p \qquad\qquad (E5)$$

Heat Absorbed in Isothermal Compression

E_A = internal energy at A

E_B = internal energy at B

$\quad$ = $E_A + dQ + dW$

dQ = heat absorbed by sample in going

$\quad\quad$ from A to B isothermally, <0.

dW = work performed on the sample in taking it from A to B isothermally, >0.

E = $E(T, \epsilon_x, \gamma^e)$

E_B = $E_A + (\partial E/\partial \epsilon_x)_{T,\gamma^e} \, d\epsilon_x + (\partial E/\partial \gamma^e)_{T,\epsilon_x} \, d\gamma^e$

From Eq. (47)

$$dE = TdS - Vpd\epsilon_x + 8V\tau d\gamma^e/3$$

$$dA = -SdT - Vpd\epsilon_x + 8V\tau d\gamma^e/3$$

Since $d\epsilon_x = dV/V$, these can also be written

$$dE = TdS - pdV + 8V\tau d\gamma^e/3$$

$$dA = -SdT - pdV + 8V\tau d\gamma^e/3$$

The elastic work consists of two terms: one arising from density change and one from elastic shear. Then

$$E_B = E_A + (\partial E/\partial V)_{T,\gamma^e} \, dV + (\partial E/\partial \gamma^e)_{T,V} \, d\gamma^e$$

$$(\partial E/\partial V)_{T,\gamma^e} = (\partial E/\partial V)_{S,\gamma^e} + (\partial E/\partial S)_{V,\gamma^e} (\partial S/\partial V)_{T,\gamma^e}$$

$$= -p + T (\partial S/\partial V)_{T,\gamma^e}$$
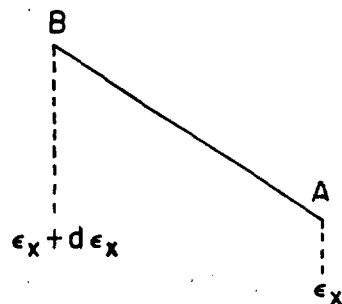
$$S = -(\partial A/\partial T)_{V,\gamma^e} \quad , \quad p = -(\partial A/\partial V)_{T,\gamma^e}$$

$$(\partial S/\partial V)_{T,\gamma^e} = -(\partial^2 A/\partial V\partial T)_{\gamma^e} = (\partial p/\partial T)_{V,\gamma^e}$$

Similarly,

$$(\partial E/\partial\gamma^e)_{T,V} = (\partial E/\partial\gamma^e)_{S,V} + (\partial E/\partial S)_{V,\gamma^e} (\partial S/\partial\gamma^e)_{T,V}$$

$$= 8V\tau/3 + T(\partial S/\partial\gamma^e)_{T,V}$$

$$(\partial S/\partial\gamma^e)_{T,V} = -(\partial^2 A/\partial T\partial\gamma^e)_V = -(8/3)(\partial V\tau/\partial T)_{V,\gamma^e}$$

$$= -(8V/3)(\partial\tau/\partial T)_{V,\gamma^e}$$

Then

$$dQ = E_B - E_A - dW = [-p + T(\partial p/\partial T)_{V,\gamma^e}] dV$$

$$+ [8V\tau/3 - (8VT/3)(\partial\tau/\partial T)_{V,\gamma^e}] d\gamma^e$$

$$+ pdV - 8V\tau d\gamma^e/3 - 8V\tau d\gamma^p/3$$

$$dQ = T(\partial p/\partial T)_{V,\gamma^e} dV - (8VT/3)(\partial\tau/\partial T)_{V,\gamma^e} d\gamma^e - 8V\tau d\gamma^p/3$$

where $8V\tau d\gamma^p/3 = dW_p$.

Evaluation of $(\partial\sigma_x/\partial T)_{\varepsilon_x^e, \varepsilon_y^e}$ and $(\partial\tau/\partial T)_{\gamma^e}$

From Eq. (27)

$$d\sigma_x = (\lambda + 2\mu) \, d\varepsilon_x^e + 2\lambda \, d\varepsilon_y^e$$

where λ and μ are functions of temperature and strain.

Let $\lambda_2 = \lambda(T_2, \varepsilon)$

$\lambda_1 = \lambda(T_1, \varepsilon)$

$\mu_2 = \mu(T_2, \varepsilon)$

$\mu_1 = \mu(T_1, \varepsilon)$

In the neighborhood of a fixed pair of elastic strains, $(\varepsilon_{xo1}^e, \varepsilon_{yo1}^e)$, σ_x can be

represented as a linear function of ε_x^e, ε_y^e:

$$\sigma_x(T_1, \varepsilon_x^e, \varepsilon_y^e) = \sigma_x(T_1, \varepsilon_{xo1}^e, \varepsilon_{yo1}^e)$$

$$+ (\lambda_1 + 2\mu_1)(\varepsilon_x^e - \varepsilon_{xo1}^e) + 2\lambda_1(\varepsilon_y^e - \varepsilon_{yo1}^e)$$

Similarly

$$\sigma_x(T_2, \varepsilon_x^e, \varepsilon_y^e) = \sigma_x(T_2, \varepsilon_{xo2}^e, \varepsilon_{yo2}^e) + (\lambda_2 + 2\mu_2)(\varepsilon_x^e - \varepsilon_{xo2}^e)$$

$$+ 2\lambda_2(\varepsilon_y^e - \varepsilon_{yo2}^e).$$

Choose ε_{xo2}^e and ε_{yo2}^e so that $\sigma_x(T_2, \varepsilon_{xo2}^e, \varepsilon_{yo2}^e) = \sigma_x(T_1, \varepsilon_{xo1}^e, \varepsilon_{yo1}^e)$. Then

$$\delta\sigma_x = (\lambda_2 + 2\mu_2)(\varepsilon_x^e - \varepsilon_{xo2}^e) + 2\lambda_2(\varepsilon_y^e - \varepsilon_{yo2}^e)$$

$$- (\lambda_1 + 2\mu_1)(\varepsilon_x^e - \varepsilon_{xo1}^e) - 2\lambda_1(\varepsilon_y^e - \varepsilon_{yo1}^e)$$

where $\delta\sigma_x \equiv \sigma_x(T_2, \varepsilon_x^e, \varepsilon_y^e) - \sigma_x(T_1, \varepsilon_x^e, \varepsilon_y^e)$.

We have approximately

$$\lambda_2 = \lambda_1 + (\partial\lambda/\partial T)\delta T, \qquad \mu_2 = \mu_1 + (\partial\mu/\partial T)\delta T$$

$$\varepsilon_{xo2}^e = \varepsilon_{xo1}^e + \alpha_x \, \delta T$$

$$\varepsilon_{yo2}^e = \varepsilon_{yo1}^e + \alpha_y \, \delta T$$

where α_x and α_y are linear thermal expansion coefficients:

$$\alpha_x = (\partial\varepsilon_x^e/\partial T)_{\varepsilon_y^e, \, \sigma_x}$$

$$\alpha_y = (\partial\varepsilon_y^e/\partial T)_{\varepsilon_x^e, \, \sigma_x}$$

Then to first order in δT

$$\delta\sigma_x = -\alpha_x(\lambda_1 + 2\mu_1) \, \delta T - 2\lambda_1 \, \alpha_y \, \delta T$$

$$+ \, \delta T(\varepsilon_x^e - \varepsilon_{xo1}^e) \, \partial(\lambda + 2\mu) \, / \, \partial T$$

$$+ \, 2\delta T \, (\varepsilon_y^e - \varepsilon_{yo1}^e) \, (\partial\lambda/\partial T)$$

$$(\partial\sigma_x/\partial T)_{\varepsilon_x^e, \varepsilon_y^e} = \lim_{\substack{\varepsilon_x^e \to \varepsilon_{xo1}^e \\ \varepsilon_y^e \to \varepsilon_{yo1}^e}} \qquad (\delta\sigma_x/\delta T) = - \, \alpha_x(\lambda + 2\mu) - 2\lambda_1 \, \alpha_y$$

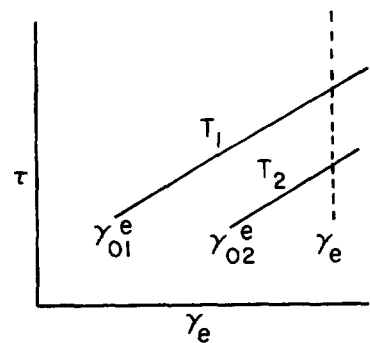If $\alpha_x = \alpha_y = \alpha/3$, where α is the volume expansion coefficient, then

$$(\partial\sigma_x/\partial T)_{\varepsilon_x^e, \varepsilon_y^e} = -\alpha(\lambda + 2\mu/3) = - \, K\alpha$$

$$= -(\partial p/\partial T)_V$$

104

Let $d\tau = 2\mu d\gamma^e$

Let $\tau(T_1, \gamma^e) = \tau(T_1, \gamma_{01}^e) + 2\mu(T_1)(\gamma^e - \gamma_{01}^e)$

$\tau(T_2, \gamma^e) = \tau(T_2, \gamma_{02}^e) + 2\mu(T_2)(\gamma^e - \gamma_{02}^e)$

Let $\tau(T_2, \gamma_{02}^e) = \tau(T_1, \gamma_{01}^e)$

Then

$$\tau(T_2, \gamma^e) - \tau(T_1, \gamma^e) \equiv \delta\tau = 2\mu(T_2)(\gamma^e - \gamma_{02}^e) - 2\mu(T_1)(\gamma^e - \gamma_{01}^e)$$

Let $\mu(T_2) = \mu(T_1) + (dT/d\mu)(T_2 - T_1)$.

Then

$$\delta\tau = -2\mu(T_1)(\gamma_{02}^e - \gamma_{01}^e) + 2(T_2 - T_1)(dT/d\mu)(\gamma^e - \gamma_{02}^e).$$

Let $\delta T \equiv T_2 - T_1 \to 0$, $\gamma_{02}^e \to \gamma_{01}^e$, $\gamma^e \to \gamma_{02}^e$

Then

$$\delta\tau/\delta T \to (\partial\tau/\partial T)_{\gamma^e} = -2\mu(T_1)(\partial\gamma^e/\partial T)_\tau$$

$\gamma^e = $ const. implies $d\gamma^e = d(\varepsilon_x^e - d\varepsilon_y^e)/2 = 0$.

105

APPENDIX H

Multiplication Time for Dislocations

Suppose that each dislocation runs freely between imperfections without multiplying, reaches an imperfection, hesitates, multiplies and goes on to the next. If the distance between imperfections is ℓ and shear velocity in the crystal is v_s, the multiplication time is the order of

$$t_0 = \ell/v_s + t_h$$

where t_h = hesitation time at the imperfection.

Let t_h be determined by a rate process. A dislocation is trapped at a site and must surmount an energy barrier, U, to escape. It makes n escape tries per second and the probability that any one of them will succeed is assumed to be exp(-U/kT). Let P(M + 1) = probability that escape occurs on the $(M + 1)^{th}$ trial. Then

$$P(M + 1) = [1 - \exp(-U/kT)]^M \exp(-U/kT)$$

$$= \text{(probability that first M tries fail)} \times$$

$$\text{(probability that } (M + 1)^{th} \text{ succeeds).}$$

Also
$$P(M + J) = P(M + 1) [1 - \exp(-U/kT)]^{J-1}.$$

The probability that escape occurs on the $(M + 1)^{th}$, $(M + 2)^{th}$, ..., or $(M + J)^{th}$ try is

$$\sum_{r=1}^{J} P(M + r) = P(M + 1)(1 - a^J)/(1 - a)$$

where $a \equiv 1 - \exp(-U/kT)$.

Since there are n tries per second, M = nt. Let J = ndt, then

$$1 - a^k = -ndt \ \ell n [1 - \exp(-U/kT)] + O(\overline{ndt}^2)$$

Neglecting the higher order quantities, we have the result that

$$P(t)dt = -[1 - \exp(-U/kT)]^{nt} \ln[1 - \exp(-U/kT)] ndt$$

where $P(t)dt$ = probability that escape occurs during the interval $(t, t + dt)$. It is readily verified that

$$\int_0^\infty P(t)\, dt = 1$$

Then the mean time for escape is

$$t_h = \int_0^\infty tP(t)dt$$
$$= -(1/n) / \ln[1 - \exp(-U/kT)]$$
$$\simeq \exp(U/kT)/n \quad \text{if } \exp(-U/kT) \ll 1.$$

The multiplication time is then

$$t_0 = \ell/v_s - (1/n) / \ln[1 - \exp(-U/kT)] \qquad \text{(H1)}$$

for $\exp(-U/kT) < 1$. The effect of applied stress is to diminish U, e.g.

$$U = U_0 (1 - \sigma/\sigma_0)$$

where σ_0 is a critical or threshold stress. For $\sigma \geq \sigma_0$, Eq. (H1) is replaced by

$$t_0 = \ell/v_s.$$

APPENDIX I
Functional dependence of p and μ

$$\partial^2 p/\partial T \partial \gamma^e = -(8/3) \; \partial(V \partial \tau/\partial T)/\partial V \tag{86a}$$

$$\partial^2 p/\partial N \partial T = -\partial^2 r/\partial V \partial T \tag{86b}$$

$$\partial^2 r/\partial T \partial \gamma^e = (8/3) \; \partial(V \partial \tau/\partial T)/\partial N \tag{86c}$$

Let $\tau = 2\mu\gamma^e$ \hfill (I1)

$$\partial\mu/\partial\gamma^e = 0 \tag{I2}$$

Let $r = b^2 \mu V/2$ \hfill (I3)

Eq. (86a), with (I1) becomes

$$\partial^2 p/\partial T \partial \gamma^e = -(16\gamma^e/3) \; \partial(V \partial\mu/\partial T)/\partial V$$

$$= -(16\gamma^e/3) \; \partial^2(\mu V)/\partial T \partial V \tag{I4}$$

If

$$\partial p/\partial \gamma^e = 0 \tag{I5}$$

Then Eq. (I4) becomes

$$\partial^2(\mu V)/\partial T \partial V = 0 \tag{I6}$$

or $\quad \partial(\mu V)/\partial T = f_1(T,\gamma^e,N)$ \hfill (I7)

In view of Eq. (I2), (I7) becomes

$$\partial(\mu V)/\partial T = f_1(T,N). \tag{I8}$$

From Eqs. (86b) and (I3),

$$\partial^2 p/\partial N \partial T = -(b^2/2) \; \partial^2 \mu V/\partial V \partial T \tag{I9}$$

109

Equations (I9) and (I6) combine to yield

$$\partial^2 p / \partial N \partial T = 0. \tag{I10}$$

Equation (I10) implies that

$$\partial p / \partial T = f_2(T, \gamma^e, V) \tag{I11}$$

Since p is already assumed to be independent of γ^e, Eq. (I11) becomes

$$\partial p / \partial T = f_2(T, V). \tag{I12}$$

Equations (86c), (I1) and (I3) combine to give

$$(b^2/2) \ \partial^2(\mu V)/\partial T \partial \gamma^e = (16\gamma^e/3) \ \partial^2(\mu V)/\partial T \partial N \tag{I13}$$

By Eq. (I2) μ is independent of γ^e, so the left hand side of Eq. (I13) is zero. Then

$$\partial^2(\mu V)/\partial T \partial N = 0 \tag{I14}$$

so,

$$\partial(\mu V)/\partial T = f_3(T, V, \gamma^e) \tag{I15}$$

Comparing Eqs. (I8) and (I15) shows that

$$\partial(\mu V)/\partial T = f_1(T) \tag{I16}$$

Integration of Eqs. (I12) and (I16) yields

$$p = f(T, V) + g(V, N) \tag{I17}$$

$$\mu V = F(T) + G(V, N) \tag{I18}$$

where the assumptions that p and μ are independent of γ^e have been carried forward into Eqs. (I17) and (I18).

110

APPENDIX J

Temperature Dependence of Shear Modulus of Al

Equation (75) is obtained from Eqs. (70) under the assumptions that pressure and specific heat, C_{V_γ}, are both independent of γ^e, and that τ is proportional to γ^e:

$$\mu = \mu_0 + b(T/V - T_0/V_0) + \phi(V) \qquad (J1)$$

Such a linear dependence is quite reasonable for small deviations from T_0, but must clearly be inadequate as T approaches the melting temperature, T_m. If one assumes that the reduction of μ with temperature is an activated process and that μ vanishes at $T = T_m$, he obtains an equation of the form

$$\mu = c \exp[-A/(T_m - T)] \qquad (J2)$$

This function varies slowly with T until it approaches T_m, then decreases precipitously in value. To give it the linear form of Eq. (J1), for small $T - T_0$, let μ be given by the expression

$$\mu = c \exp[-A/(T_m - T)] [1 - b(T - T_0)] \qquad (J3)$$

Measurements made by P. M. Sutton (1953) have been used to evaluate the constants in Eq. (J3) with $T_0 = 273.2°K$. The result is

$$\mu(\text{megabars}) = .2812 \exp[-7.8059/(933.2 - T)]$$
$$[1 - 4.79 \times 10^{-4} (T - 273.2)] \qquad (J4)$$

Values calculated from Eq. (J4) are compared with Sutton's measured values in the following table:

T	=	273.2	373.2	473.2	573.2	673.2	773.2	873.2	903.2	933.2
μ_{calc}	=	.2812	.2672	.2530	.2384	.2232	.2061	.1780	.153	0
μ_{meas}	=	.2812	.2670	.2530	.2385	.2232	.2063			

These values are plotted in Fig. J1.

It would improve the form of (J3) to write it as

$$\mu = c \exp[-A/(T_m - T)] [1 - B'(T/V - T_o/V_o)] \qquad (J5)$$

In that case

$$b' \simeq 1.8 \times 10^{-4} \text{ cc/g}°K .$$

However, one can argue that the parameter A should also be volume dependent, so the refinement of (J5) hardly seems worthwhile at present.

A different form of the temperature dependence of elastic constants has been given by Varshni (1970). It describes the slow variation satisfactorily but does not vanish at $T = T_m$. For the present purpose the behavior near $T = T_m$ is important because it relates to the annealing properties of dislocations.
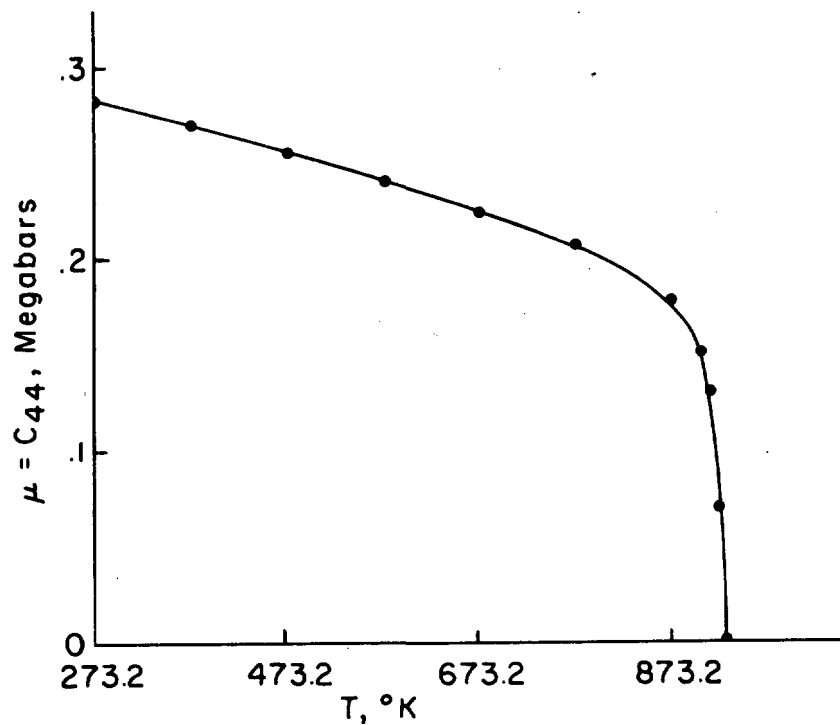
Figure J1 C_{44} of Al according to the formula

$$C_{44} = .2812 \exp[-7.8059/(933.2-T]$$
$$[1 - 4.79 \times 10^{-4} (T-273.2)]$$

This corresponds closely to data given by P. M. Sutton (1953).

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119

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DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*	2a. REPORT SECURITY CLASSIFICATION
Washington State University	UNCLASSIFIED
Department of Physics	2b. GROUP
Pullman, Wa. 99163	

3. REPORT TITLE

THEORY OF EQUATIONS OF STATE: ELASTIC-PLASTIC EFFECTS II

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Scientific   Final

5. AUTHOR(S) *(First name, middle initial, last name)*

George E. Duvall and D. P. Dandekar

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
MAY 1973	123	73

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
DAAD05-71-C-0136	
b. PROJECT NO.	BRL CONTRACT REPORT NO. 106
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	WSU-SDL 72-01

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13. ABSTRACT

After a brief review of the history of material failure studies and of measurements of stored energy in plastically deformed bodies, energy and stress and strain relations are developed for both elastic and plastic deformation in uniaxial strain. The physical effects of plastic deformation are discussed and foundations are laid for thermodynamic calculations of plastically deformed materials. Calculations of both thermodynamic and mechanical parameters are described for uniaxial strain with and without energy storage in internal strains. Conditions of uniaxial strain are found to yield new thermodynamic relations which are useful for synthesizing constitutive relations.

DD FORM 1 NOV 65 **1473**